Two-Loop Tensor Integrals in Quantum Field Theory*

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A comprehensive study is performed of general massive, tensor, two-loop Feynman diagrams with two and three external legs. Reduction to generalized scalar functions is discussed. Integral representations, supporting the same class of smoothness algorithms already employed for the numerical evaluation of ordinary scalar functions, are introduced for each family of diagrams.

Key words: Feynman diagrams, Multi-loop calculations, Self-energy Diagrams, Vertex diagrams

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1 Introduction

This paper is the fifth in a series devoted to the numerical evaluation of multi-loop, multi-leg Feynman diagrams. In [1] (hereafter I) the general strategy was outlined and in [2] (hereafter II) a complete list of results was derived for two-loop functions with two external legs, including their infrared divergent on-shell derivatives. Results for one-loop multi-leg diagrams were shown in [3] and additional material can be found in [4]. Two-loop three-point functions for infrared convergent configurations were considered in [5] (hereafter III), where numerical results can be found.

In this article we study the problem of deriving a judicious and efficient way to deal with tensor Feynman integrals, namely those integrals that occur in any field theory with spin and non trivial structures for the numerators of Feynman propagators. Admittedly the topic of this paper is rather technical, but it is needed as a basis for any realistic calculation of physical observables at the two-loop level.

The complexity of handling two-loop tensor integrals is reflected in the following simple consideration: the complete treatment of one-loop tensor integrals was confined to the appendices of [6], while the reduction of general two-loop self-energies to standard scalar integrals already required a considerable fraction of [7]; the inclusion of two-loop vertices requires the whole content of this paper. Past experience in the field has shown the convenience of gathering a complete collection of results needed for a broad spectrum of applications in one place. We devote the present article to this task.

While a considerable amount of literature is devoted to the evaluation of two-loop scalar vertices [8], fewer papers deal with the tensor ones [9]; for earlier attempts to reduce and evaluate two-loop graphs with arbitrary masses we refer the reader to the work of Ghinculov and Yao [10].

In recent years, the most popular and quite successful tool in dealing with multi-loop Feynman diagrams in QED/QCD (or in selected problems in different models, characterized by a very small number of scales), has been the Integration-By-Parts Identities (IBPI) method [11]. However, reduction to a set of Master Integrals (MI) is poorly known in the enlarged scenario of multi-scale electroweak physics.

Our experience with one-loop multi-leg diagrams [3] shows that the optimal algorithm to deal with realistic calculations should be able to treat both scalar and tensor integrals on the same footing. This algorithm should not introduce multiplications of the tensor integrals by negative powers of Gram determinants, as the latter's zeros, although unphysical, may be dangerously close to the physically allowed region. The numerical quality of tensor integrals also worsens if they are expressed in terms of linear combinations of MI; the coefficients of these combinations have zeros corresponding to real singularities of the diagram [12], and the singular behavior is usually badly overestimated leading to numerical instabilities.

From the point of view of numerical integration, it really makes little difference if tensor integrals are expressed in terms of generalized scalar configurations, or in terms of smooth integral representations which do not grant any privilege to a particular member of the same class of integrals. Of course, at the end of the day we are always left with the problem of numerical cancellations (an issue related to the strategy of trading one difficult integral for many simpler ones), and the optimal algorithm should minimize the number of smooth integrals in the final answer. There is no evidence that employing our approach one encounters more objectionable features than in reducing everything to MI; rather, in our opinion, the feasibility of the latter has still to be proved in the complex environment of the full-fledged Standard Model (SM), even if there are complete applications in QED [13] and in QCD [14].

We have not included four-point functions in the classification, although they are certainly needed to compute physical observables for fermion–anti-fermion annihilations or scattering processes, not yet a top priority in handling electroweak radiative corrections in the SM at the two-loop level. Note, however, that there is intense activity in (QED) QCD scattering processes [14] and [15]. Addressing the full set of corrections is, by necessity, a long term project which we undertake step-by-step (an attitude which should not be confused with narrow focusing).

A large fraction of physical processes, in particular gauge bosons decays into fermion–anti-fermion pairs and accurate predictions for gauge boson complex poles, only require two- and three-point functions. Also for the analysis of the two-loop SM renormalization [16], two–point functions and vacuum bubbles are essentially all we need. Indeed, in order to evaluate the Fermi coupling constant G_F from the muon life-time we always work at zero momentum transfer and neglect terms proportional to m_μ^2/M_W^2; therefore, all diagrams contributing to this process (boxes included) are simply equivalent to vacuum bubbles, i.e. generalized sunset integrals evaluated at zero external momentum.

Feynman diagrams are built using propagators and vertices. In momentum space, the former are represented by

$$\frac{N(p)}{p^2+m^2-i\,\delta}\;, \tag{1}$$

where $\delta\to 0_+$, m is the bare mass of the particle and $N(p)$ is an expression depending on its spin. Our general approach towards the numerical evaluation of an arbitrary, multi-scale, Feynman diagram G is to use a Feynman parameter representation and to obtain, diagram-by-diagram, an integral representation of the following form:

$$G=\sum\left[\frac{1}{B_{\scriptscriptstyle G}}\int_{\scriptscriptstyle S} dx\,\mathcal{G}(x)\right], \tag{2}$$

where x is a vector of Feynman parameters, S is a simplex, $\mathcal{G}$ is an integrable function (in the limit $\delta\to 0_+$) and $B_{\scriptscriptstyle G}$ is a function of masses and external momenta whose zeros correspond to true singularities of the diagram G, if any. The Bernstein-Tkachov (hereafter BT) functional relations [17] are one realization of Eq.(2), but in our previous work we considered different possibilities.

Smoothness requires that the kernel in Eq.(2), together with its first N derivatives, should be a continuous function, with N as large as possible. However, in most cases we will be satisfied with absolute convergence, e.g. with logarithmic singularities of the kernel. This is particularly true around the zeros of $B_{\scriptscriptstyle G}$, where the large number of terms, induced requiring continuous derivatives of higher orders, leads to large numerical cancellations.

As we stressed earlier, this article is by its own nature rather technical, but we tried to avoid as much as possible a layout which overwhelmingly privileges long lists of formulae in favor of interleaving the indispensable amount of technical details with examples. For completeness, however, we inserted Appendices where the reader can find a complete summary of the results occurring in the reduction procedure.

The results presented in this paper are intermediate steps in any physical calculation; although the presentation is organized through a series of concatenated formulae that can be used recursively, further derivations on the part of the reader are required in order to obtain analytic or numerical results for a physical quantity.

The outline of the paper is as follows: in Section 2 we recall our notation and conventions. In Section 3 we review the problem of gauge cancellations and the use of Nielsen identities, while in Section 4 we illustrate all preliminary steps that should be undertaken in any realistic calculation (like projector techniques). The reduction of two-loop two-point functions is discussed in Section 5 and a complete list of the results is given in Section 7. The role of integration-by-part identities is discussed in Section 6. In Sections 8–11 we present the full body of our results for two-loop tensor integrals. (Rank three tensors for three-point functions are shown in Section 9.7.) Conclusions are drawn in Section 12. Additional material is discussed in the Appendices; in particular, the treatment of generalized one-loop functions is discussed in Appendix A. A concatenated set of easy-to-use formulae for the reduction of two-loop three-point functions is summarized in Appendix B; symmetries of diagrams are presented in Appendix C.

2 Conventions and notation

Our conventions for arbitrary two-loop diagrams were introduced in Sect. 2 of I. Specific conventions for three-point functions were introduced in Sect. 2 of III; vertex topologies were classified in III and are reproduced, for the reader's convenience, in Figs. 6–11. Here we briefly recall the terminology.

A generalized one-loop diagram will be denoted by

$$G_{\mu_1\,,\,\cdots\,,\,\mu_L}(\{\alpha\}_{\scriptscriptstyle N}\,;\,\{p\}_{\scriptscriptstyle N-1}\,,\,\{m\}_{\scriptscriptstyle N})=\frac{\mu^{4-n}}{i\,\pi^2}\int d^nq\,q_{\mu_1}\,\cdots\,q_{\mu_L}\,\prod_{i=1}^{N}[i]_{\scriptscriptstyle G}^{-\alpha_i}, \tag{3}$$

where $n = 4 - \epsilon$, n is the space-time dimension,[1] μ is the arbitrary unit of mass, N is the number of vertices, and

$$\{\alpha\}_N = \alpha_1\,,\,\cdots\,,\,\alpha_N\,, \qquad [i]_G = (q + \sum_{j=0}^{i-1} p_j)^2 + m_i^2, \quad p_0 = 0\,. \tag{4}$$

The one-loop two-, three-,... point functions will be denoted by $G = B, C, \cdots$.

A generalized two-loop diagram is defined with arbitrary, non-canonical, powers of its propagators; it can be cast in the following form

$$G^{\{\alpha\}_a\,|\,\{\beta\}_b\,|\,\{\gamma\}_c}(\mu_1\,,\,\cdots\,,\,\mu_R\,|\,\nu_1\,,\,\cdots\,,\,\nu_S\,;\,\{\eta^1\,p\}\,,\,\{\eta^{12}\,p\}\,,\,\{\eta^2\,p\}\,,\,\{m\}_{a+b+c}) =$$
$$\frac{\mu^{2(4-n)}}{\pi^4} \int d^nq_1\, d^nq_2 \prod_{r=1}^{R} q_{1\mu_r} \prod_{s=1}^{S} q_{2\nu_s} \prod_{i=1}^{a} (k_i^2 + m_i^2)^{-\alpha_i} \prod_{j=a+1}^{a+c} (k_j^2 + m_j^2)^{-\gamma_j} \prod_{l=a+c+1}^{a+c+b} (k_l^2 + m_l^2)^{-\beta_l}, \tag{5}$$

where a, b and c indicate the number of lines in the q_1, q_2 and $q_1 - q_2$ loops, respectively. For generalized functions we use $\alpha = \sum_{i=1}^{a} \alpha_i$ etc, while for standard functions (i.e. those where all the propagators have canonical power -1), $\alpha = a, \beta = b$ and $\gamma = c$ and we will write $G^{\alpha\beta\gamma}$. Furthermore,

$$\begin{array}{ll} k_i = q_1 + \sum_{j=1}^{N} \eta^1_{ij}\, p_j\,, & i = 1, \ldots, a\,, \\ k_i = q_1 - q_2 + \sum_{j=1}^{N} \eta^{12}_{ij}\, p_j\,, & i = a+1, \ldots, a+c\,, \\ k_i = q_2 + \sum_{j=1}^{N} \eta^2_{ij}\, p_j\,, & i = a+c+1, \ldots, a+c+b\,, \end{array}$$

where $\eta^s = \pm 1$, or 0, and $\{p\}$ is the set of external momenta. Diagrams which can be reduced to combinations of other diagrams with a smaller number of internal lines will not receive a particular name. Otherwise, a two-loop diagram will be denoted by $G^{\alpha\beta\gamma}$, where $G = S, V, B$ etc. stands for two-, three-, four-point etc. For scalar integrals we will use the symbol $G_0^{\alpha\beta\gamma} = G^{\alpha\beta\gamma}(0|0;\ldots)$. Following Eq.(5) diagrams are further classified according to non empty entries in the matrices η^s and in the list of internal masses.

Integrals: To keep our results as compact as possible, we introduce the following notation ($x_0 = y_0 = 1$) where C stands for (hyper)cube and S for simplex,

$$\int dCS\,(\{x\}\,;\,\{y\})\, f(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}) = \int_0^1 \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} \int_0^{y_{j-1}} dy_j f(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}),$$
$$\int dS_n(\{x\})\, f(x_1, \cdots, x_n) = \prod_{i=1}^{n} \int_0^{x_{i-1}} dx_i\, f(x_1, \cdots, x_n),$$
$$\int dC_n(\{x\}) f(x_1, \cdots, x_n) = \int_0^1 \prod_{i=1}^{n} dx_i f(x_1, \cdots, x_n). \tag{6}$$

Also, the so-called $'+'$-distribution will be extensively used, e.g.

$$\int dC_n(\{z\}) \int_0^1 dx\, \frac{f(x,\{z\})}{x}\Big|_+ = \int dC_n(\{z\}) \int_0^1 dx\, \frac{f(x,\{z\}) - f(0,\{z\})}{x},$$
$$\int dC_n(\{z\}) \int_0^1 dx\, \frac{f(x,\{z\})}{x-1}\Big|_+ = \int dC_n(\{z\}) \int_0^1 dx\, \frac{f(x,\{z\}) - f(1,\{z\})}{x-1},$$
$$\int dC_n(\{z\}) \int_0^1 dx\, \frac{f(x,\{z\}) \ln^n x}{x}\Big|_+ = \int dC_n(\{z\}) \int_0^1 dx\, \frac{\Big[f(x,\{z\}) - f(0,\{z\})\Big] \ln^n x}{x}. \tag{7}$$

The last relation in Eq.(7) is used when evaluating integrals of the following type:

$$\int_0^1 dx\, \frac{f(x)}{x^{1-\epsilon}} = \frac{f(0)}{\epsilon} + \int_0^1 dx\, \frac{f(x)}{x}\Big|_+ - \epsilon \int_0^1 dx\, \frac{f(x)\, \ln x}{x}\Big|_+ + \mathcal{O}\left(\epsilon^2\right).$$

[1] In our metric, space-like p implies $p^2 = \vec{p}^{\,2} + p_4^2 > 0$. Also, it is $p_4 = i\, p_0$ with p_0 real for a physical four-momentum.

Lists of arguments: To avoid long lists of arguments we introduce the symbol

$$\{m\}_{i\,j\,\cdots\,k} = m_i\,,\,m_j\,,\,\cdots\,,\,m_k, \qquad \text{exactly in this order.} \tag{8}$$

Miscellanea: We often need combinations of squared masses and momenta,

$$\begin{aligned} l_{ijk} &= p_i^2 - m_j^2 + m_k^2, \qquad l_{Pjk} = P^2 - m_j^2 + m_k^2, \qquad l_{pjk} = p^2 - m_j^2 + m_k^2, \\ m_{ijk}^2 &= m_i^2 - m_j^2 + m_k^2, \qquad m_{ij}^2 = m_i^2 - m_j^2, \qquad p_{ij} = p_i \cdot p_j, \\ (G)_{ij} &= p_{ij}, \quad D = \det G = p_1^2\,p_2^2 - (p_1 \cdot p_2)^2, \quad D_1 = p_1^2\,p_2^2, \quad D_2 = p_{12}\,p_2^2, \quad D_3 = p_{12}\,p_1^2, \end{aligned} \tag{9}$$

and of Feynman parameters,

$$\overline{x} = 1 - x, \qquad \overline{x}_i = 1 - x_i, \qquad \overline{y}_i = 1 - y_i, \quad \text{etc}, \quad X = \frac{1 - x_1}{1 - x_2} = 1 - \overline{X} \tag{10}$$

$$Y_i = -(1 - y_i + y_3\,X)\,, \quad \overline{Y}_2 = 1 - y_2\,\overline{X}, \quad H_i = 1 - x_1 - x_2\,Y_i\,, \quad i = 1, 2\,. \tag{11}$$

$$F(x,y) = p_1^2\,x^2 + 2\,p_{12}\,x\,y + p_2^2\,y^2, \qquad m_x^2 = \frac{m_1^2}{x} + \frac{m_2^2}{1-x}. \tag{12}$$

Symmetrized tensors: We define (partially) symmetrized tensors as follows ($\delta_{\alpha\beta}$ is the Kronecker delta function),

$$\begin{aligned} \{p\,k\}_{\mu\nu} &= p_\mu\,k_\nu + p_\nu\,k_\mu, \qquad \{\delta\,p\}_{\alpha\beta\gamma} = \delta_{\alpha\beta}p_\gamma + \delta_{\alpha\gamma}p_\beta + \delta_{\beta\gamma}p_\alpha, \\ \{ppk\}_{\alpha\beta\gamma} &= p_\alpha\,p_\beta\,k_\gamma + p_\alpha\,p_\gamma\,k_\beta + p_\gamma\,p_\beta\,k_\alpha \\ \{ppk\}_{\alpha\gamma\,|\,\beta} &= p_\alpha p_\beta k_\gamma + p_\gamma p_\beta k_\alpha, \qquad \{\delta\,p\}_{\alpha\beta\,|\,\gamma} = \delta_{\alpha\gamma}\,p_\beta + \delta_{\beta\gamma}\,p_\alpha. \end{aligned} \tag{13}$$

Contraction: If p is a vector and f is a function, we introduce the symbol

$$f(\cdots\,,\,p\,,\,\cdots) = p^\mu\,f(\cdots\,,\,\mu\,,\,\cdots), \qquad f(\cdots\;\mu\mu\,,\,\cdots) = \delta^{\mu\nu}\,f(\cdots\;\mu\nu\,,\,\cdots). \tag{14}$$

$\overline{MS}$ factors: Finally we remind the reader of the definition of $\overline{MS}$ factors,

$$\Delta_{UV} = \gamma + \ln\pi - \ln\frac{\mu^2}{\mid P^2\mid}, \qquad \overline{\Delta}_{UV} = \frac{1}{\epsilon} - \Delta_{UV}, \qquad \omega = \frac{\mu^2}{\pi}\;, \tag{15}$$

where $\gamma = 0.577216\cdots$ is the Euler constant. In one-loop calculations the definition $\overline{\Delta}_{UV} = 2/\epsilon - \Delta_{UV}$ is often employed. Finally some authors prefer to define $n = 4 - 2\,\epsilon$.

2.1 Definition of one-loop generalized functions

Products of one-loop functions occur in the reduction of two-loop diagrams; generalized one-loop functions are defined in Eq.(3), specific examples of one- and two-point (scalar) functions are

$$\begin{aligned} A_0(\alpha; m) &= \frac{\mu^\epsilon}{i\,\pi^2}\int\frac{d^nq}{(q^2+m^2)^\alpha}\;, \qquad A_0(\alpha;[m_i,m_j]) = A_0(\alpha;m_i) - A_0(\alpha;m_j), \\ B_0(\alpha,\beta\,;\,p,m_1,m_2) &= \frac{\mu^\epsilon}{i\,\pi^2}\int\frac{d^nq}{(q^2+m_1^2)^\alpha((q+p)^2+m_2^2)^\beta}. \end{aligned} \tag{16}$$

Note that we always drop strings like $1, 1, \cdots$ in the argument of standard functions, namely, we write $A_0(m)$ for $A_0(1,m)$ etc. Tensor integrals are:

$$\frac{\mu^\epsilon}{i\,\pi^2}\int\frac{d^nq\,q_\mu}{(q^2+m_1^2)^\alpha\,((q+p)^2+m_2^2)^\beta} = B_1(\alpha,\beta\,;\,p,\{m\}_{12})\,p_\mu, \tag{17}$$

$$\frac{\mu^\epsilon}{i\,\pi^2}\int\frac{d^nq\,q_\mu q_\nu}{(q^2+m_1^2)^\alpha\,((q+p)^2+m_2^2)^\beta}=B_{21}(\alpha,\beta\,;\,p,\{m\}_{12})\,p_\mu\,p_\nu+B_{22}(\alpha,\beta\,;\,p,\{m\}_{12})\,\delta_{\mu\nu},\tag{18}$$

their reduction is given in Section 7. Generalized one-loop three-point functions are introduced as follows:

$$C_{\mu_1,\cdots\mu_l}(\{\alpha\}_3\,;\,p_1,p_2,\{m\}_{123})=\frac{\mu^\epsilon}{i\,\pi^2}\int d^nq\,\prod_{j=1}^{l}q_{\mu_j}\,\prod_{i=1}^{3}[i]^{-\alpha_i},\tag{19}$$

with $[i]=Q_i^2+m_i^2$ and $Q_i=q+p_0+\cdots+p_{i-1}$, $p_0=0$. In particular,

$$C_\mu(p_1,p_2,\{m\}_{123})=\frac{\mu^\epsilon}{i\,\pi^2}\int d^nq\frac{q_\mu}{[q^2+m_1^2][(q+p_1)^2+m_2^2][(q+p_1+p_2)^2+m_3^2]}.\tag{20}$$

The integrals of Eq.(19) can be reduced, for example,

$$C_\mu(\{\alpha\}_3\,;\,p_1,p_2,\{m\}_{123})=C_{11}(\{\alpha\}_3\,;\,\cdots)\,p_{1\mu}+C_{12}(\{\alpha\}_3\,;\,\cdots)\,p_{2\mu},\tag{21}$$

$$\begin{aligned}C_{\mu\nu}(\{\alpha\}_3\,;\,p_1,p_2,\{m\}_{123})&=C_{21}(\{\alpha\}_3\,;\,\cdots)\,p_{1\mu}p_{1\nu}+C_{22}(\{\alpha\}_3\,;\,\cdots)\,p_{2\mu}p_{2\nu}\\&+C_{23}(\{\alpha\}_3\,;\,\cdots)\,\{p_1p_2\}_{\mu\nu}+C_{24}(\{\alpha\}_3\,;\,\cdots)\,\delta_{\mu\nu},\end{aligned}\tag{22}$$

where the symmetrized product is defined by Eq.(13); for the reduction we refer to Appendix A.

For completeness we also define generalized one-loop four-point functions, although they are not needed in this article :

$$D_{\mu_1,\cdots,\mu_l}(\{\alpha\}_4\,;\,p_1,p_2,p_3,\{m\}_{1234})=\frac{\mu^\epsilon}{i\,\pi^2}\int d^nq\,\prod_{j=1}^{l}q_{\mu_j}\,\prod_{i=1}^{4}[i]^{-\alpha_i},\tag{23}$$

etc. Once again $[i]=Q_1^2+m_i^2$, $Q_i=q+p_0+\cdots+p_{i-1}$ with $p_0=0$.

2.2 Alphameric classification of graphs

In our conventions any scalar two-loop diagram is identified by a capital letter (S,V, etc.) indicating the number of external legs, and by a triplet of numbers (α,β and γ) giving the number of internal lines (carrying internal momenta q_1,q_2 and q_1-q_2, respectively). There is a compact way of representing this triplet: assume that $\gamma\neq 0$, i.e. that we are dealing with non-factorizable diagrams, then we introduce

$$\kappa=\gamma_{\max}\Big[\alpha_{\max}\,(\beta-1)+\alpha-1\Big]+\gamma\tag{24}$$

for each diagram. For $G=V$ we have $\alpha_{\max}=2$ and $\gamma_{\max}=2$. Furthermore, we can associate a letter of the English alphabet to each value of κ. Therefore, the following correspondence holds:

$$121\to E,\quad 131\to I,\quad 141\to M,\quad 221\to G,\quad 231\to K,\quad 222\to H.\tag{25}$$

For $G=S$ we have $\alpha_{\max}=2$ and $\gamma_{\max}=1$, therefore

$$111\to A,\quad 121\to C,\quad 131\to E,\quad 221\to D.\tag{26}$$

This classification is extensively used throughout the paper and motivated by the unavoidable proliferation of indices; the reader not familiar with it should remember that storing the elements of a matrix into a vector is a well-known coding procedure (e.g. in Fortran). Note that in II and in III this convention was not yet used and the correspondence of results is simply provided by Eqs.(25)–(26).

3 Tensor integrals and gauge cancellations

Any Feynman integral with a tensor structure can be written as a combination of form factors

$$G_{\mu_1 \cdots \mu_N} = \sum_{i=1}^{i_{\max}} G^i_S \, F_{i\,;\,\mu_1 \cdots \mu_N}, \tag{27}$$

where the $F_{i\,;\,\mu_1 \cdots \mu_N}$ are tensor structures made up of external momenta, Kronecker delta functions, ϵ-tensors (which will cancel in any CP-even observable) and elements of the Dirac algebra; the scalar projections G^i_S admit a parametric representation which is equivalent to the one for the corresponding scalar diagram but with polynomials of Feynman parameters occurring in the numerator. Once we have an integral representation for the primary scalar diagram, with the desired properties of smoothness, then, analogous representations, with the same properties, also follow for the induced scalar projections. Therefore, from the point of view of numerical evaluation there is really little difference between scalar and tensor integrals.

However, there is a problem due to the fact that we are dealing with gauge theories with inherent gauge cancellations which do not support a blind application of the procedure just described. A very simple example will be useful to illustrate the roots of this problem. Consider the one-loop photon self-energy in QED and express the result in terms of scalar one-loop form factors [6]; we obtain

$$\Pi^f_{\mu\nu} = \Pi^f_1 \delta_{\mu\nu} + \Pi^f_2 p_\mu p_\nu, \tag{28}$$

$$\Pi^f_1 = -4\, e^2 \Big\{ (2-n) B_{22} - p^2 \Big[B_{21} + B_1 \Big] - m^2_f \, B_0 \Big\}, \qquad \Pi^f_2 = -8\, e^2 \Big[B_{21} + B_1 \Big], \tag{29}$$

where e is the bare electric charge and B_0 etc. are the standard one-loop functions of [6], all with arguments $(p^2; m_f, m_f)$. The gauge invariance of the theory is controlled by a set of Ward–Slavnov–Taylor identities [18] (hereafter WST), one of which requires $\Pi^f_{\mu\nu}$ to be transverse; this hardly follows from expressing the form factors in parametric space followed by some numerical integration. Rather, it follows from a set of identities that one can write among the standard one-loop functions (B_{22} etc.) directly in momentum space, the so-called reduction procedure ("scalarization" in jargon). This procedure, in its original design, is plagued by the occurrence of inverse powers of Gram determinants whose zeros are unphysical but sometimes dangerous for the numerical stability of the result.

There is another example where gauge cancellations play a crucial role. Suppose that we decide to work in the so-called R_ξ gauges with one or more gauge parameters which we will collectively indicate by ξ: the expected ξ independence is seen at the level of S-matrix elements and not for individual contributions to Green functions. From this point of view, any procedure that computes single diagrams and sums the corresponding numerical results, without controlling gauge cancellations analytically, is bound to have its own troubles.

These two rather elementary considerations suggest the following strategy: first impose all the requirements dictated by WST identities and see that they are satisfied. At this point organize the calculation according to building blocks that are, by construction, gauge-parameter independent.

The first step requires some form of scalarization (which, as we saw, may be numerically unstable), but the perspective is different: scalarization is now needed only to prove that certain combinations of form factors are zero, and any occurrence of Gram determinants does not therefore pose a problem.

In the second step we need to control the ξ behavior of individual Green functions; a possible tool is represented by the use of the Nielsen identities (hereafter NI) [19]. Typically we will consider the transverse propagator of a gauge field:

$$D_{\mu\nu} = \frac{1}{(2\pi)^4\, i} \, \frac{\delta_{\mu\nu} - p_\mu p_\nu / s}{s - M^2_0 + \Pi(\xi, s)}, \tag{30}$$

where $p^2 = -s$, M_0 is the bare mass of the particle and $\Pi(\xi, s)$ is the self-energy. The corresponding NI reads as follows:

$$\frac{\partial}{\partial\, \xi}\, \Pi(\xi, s) = \Lambda(\xi, s)\, \Pi(\xi, s), \tag{31}$$

where Λ is a complex, amputated, 1PI, two-point Green function and the complex pole is defined by

$$\bar{s} - M_0^2 + \Pi(\xi, \bar{s}) = 0, \qquad \partial_\xi\, \bar{s} = 0. \tag{32}$$

Let us consider now the amplitude for $i \to V \to f$, where V is an unstable gauge boson and i/f are initial/final states. The overall amplitude becomes

$$A_{fi}(s) = \frac{\delta_{\mu\nu}}{s - \bar{s}}\, \frac{V_f^\mu(\bar{s})\, V_i^\nu(\bar{s})}{1 + \Pi'(\bar{s})} \quad + \quad \text{non-resonant terms}, \tag{33}$$

where it is understood that the vertex functions V_f^μ and V_i^ν include the wave-function renormalization factors of the external, on-shell, particles. It has been proved that

$$\frac{d}{d\,\xi} \Big[1 + \Pi'(\bar{s})\Big]^{-1/2} V_f^\mu(\bar{s}) = 0; \tag{34}$$

this combination is the prototype of one of the gauge-parameter independent building blocks that are needed to assemble our calculation of a physical observables. All gauge-parameter independent blocks will then be mapped into one (multi-dimensional) integral to be evaluated numerically.

4 Projector techniques

Any realistic calculation requires several steps to be performed before we can actually start to compute diagrams or sums of diagrams; in all of them, some action can be taken in order to simplify the structure of the amplitude in some efficient way. Much work has been invested in this area and we refer to recent work of Glover [20] for an exhaustive list of references.

Here we focus on few examples. Consider, for instance, the matrix element for the decay of a vector particle V into a fermion-antifermion pair, $V(P_+) \to \bar{f}(p_+) f(p_-)$ (all particles are on their mass-shell); instead of decomposing all tensor integrals into form factors, we first decompose the vertex V_μ into the following structures,

$$\mathcal{M} = \bar{u}(p_-)\ \epsilon \cdot V\, v(p_+) = \bar{u}(p_-)\, \Big[F_V\, \not\epsilon + F_A \not\epsilon\, \gamma_5 + F_S\, P_- \cdot \epsilon + F_P\, P_- \cdot \epsilon\, \gamma_5\Big]\, v(p_+), \tag{35}$$

where $\epsilon_\mu = \epsilon_\mu(P_+)$ is the polarization vector for the V particle, subject to the constraint $\epsilon \cdot P_+ = 0$ ($P_\pm = p_+ \pm p_-$). We also introduce projectors to extract the form factors appearing in Eq.(35) [20]

$$\sum_{\text{spin}} P_I\, \mathcal{M} = F_I, \tag{36}$$

where $I = V, A, S$ or P. The explicit solution for the projectors is obtained considering four auxiliary quantities

$$\begin{aligned} P_1 &= \bar{v}(p_+) \not\epsilon\, u(p_-), & P_2 &= \bar{v}(p_+) \not\epsilon\, \gamma_5\, u(p_-), \\ P_3 &= \epsilon \cdot P_-\, \bar{v}(p_+)\, u(p_-), & P_4 &= \epsilon \cdot P_-\, \bar{v}(p_+)\, \gamma_5\, u(p_-). \end{aligned}$$

Let us define $\beta_M = M^2 - 4\, m^2$, where M is the mass of the vector boson V and m is the mass of the fermion f: we get

$$\begin{aligned} P_V &= -\,\frac{1}{2\,(2-n)\,M^2}\, \Big[P_1 + 2\,i\,\frac{m}{\beta_M}\, P_3\Big], \qquad P_A = -\,\frac{1}{2\,(2-n)\,\beta_M}\, P_2, \\ P_P &= \frac{1}{2\,M^2\,\beta_M}\, P_4, \qquad P_S = -\,i\,\frac{m}{M^2\,(2-n)\,\beta_M}\, \Big\{P_1 + \frac{i}{2\,m\,\beta_M}\, \Big[4\,m^2 + (n-2)\,M^2\Big]\, P_3\Big\}, \end{aligned} \tag{37}$$

thus providing the scalar coefficients F_I. For example,

$$\begin{aligned} F_V = \frac{1}{n-2}\, \text{Tr}\, \mathcal{F}_V, \qquad \mathcal{F}_V &= -\ \frac{1}{2\,M^2}\, \gamma^\mu\, \Lambda_-\, V_\mu\, \Lambda_+ - \frac{i}{2\,M^4}\, \Lambda_+\, \Lambda_-\, P_+ \cdot V\, \Lambda_+ \\ &\quad -\ \frac{i\,m}{M^4}\, \Lambda_-\, P_+ \cdot V\, \Lambda_+ - \frac{i\,m}{M^2\,\beta_M}\, \Lambda_-\, P_- \cdot V\, \Lambda_+, \end{aligned} \tag{38}$$

where $\Lambda_+ = -i\,\not{p}_+ - m$ and $\Lambda_- = -i\,\not{p}_- + m$. This procedure completely saturates indices and allows us to consider only integrals with positive powers of scalar products in the numerators. Then a reduction procedure follows and we will show that the final answer contains only generalized scalar integrals. For a discussion on projector techniques in conventional dimensional regularization or in the 't Hooft-Veltman scheme [21] we refer again to [20].

Another example we want to consider is the amplitude for $s \to \gamma\gamma$, where s is a generic scalar particle; for this case we follow the procedure of Binoth, Guillet and Heinrich [22] and introduce the vectors

$$r_{i\mu} = \sum_{j=1}^{i} p_{j\mu}, \qquad R_{i\mu} = \sum_{j=1}^{2} G^{-1}_{ij}\, r_{j\mu}, \quad G_{ij} = 2\, r_i \cdot r_j. \tag{39}$$

The square of the $s \to \gamma\gamma$ vertex is further decomposed into

$$V_{\mu\nu} = F_{\scriptscriptstyle D}\, \delta_{\mu\nu} + \sum_{i,j=1}^{2} F_{\scriptscriptstyle P,ij}\, r_{i\mu}\, r_{j\nu}. \tag{40}$$

The form factors of this decomposition are expressed through the action of projectors,

$$F_{\scriptscriptstyle D} = P^{\mu\nu}_{\scriptscriptstyle D}\, V_{\mu\nu}, \qquad F_{\scriptscriptstyle P,ij} = P^{\mu\nu}_{\scriptscriptstyle P,ij}\, V_{\mu\nu}, \tag{41}$$

$$P^{\mu\nu}_{\scriptscriptstyle D} = \frac{1}{n-2}\Bigl[\delta^{\mu\nu} - 2\, r^\mu\, G^{-1}\, r^\nu\Bigr], \qquad P^{\mu\nu}_{\scriptscriptstyle P,ij} = 4\Bigl[R^\mu_i\, R^\nu_j - \frac{1}{2}\, G^{-1}_{ij}\, P^{\mu\nu}_{\scriptscriptstyle D}\Bigr]. \tag{42}$$

These projectors have the following properties:

$$R_i \cdot r_j = \frac{1}{2}\delta_{ij}, \qquad P^{\mu\nu}_{\scriptscriptstyle D}\, r_{i\nu} = 0, \qquad P^{\mu\nu}_{\scriptscriptstyle D}\, P_{\scriptscriptstyle D,\mu\nu} = \frac{1}{n-2}. \tag{43}$$

The whole procedure is better illustrated in terms of an example, an I-family contribution to the decay $H \to \gamma\gamma$, see Fig. 1. The corresponding integral is

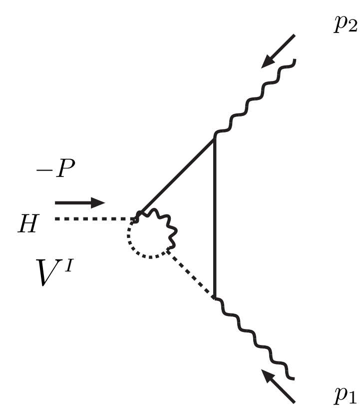


Figure 1: I-family contribution to the decay $H \to \gamma\gamma$. Internal dotted lines represent a Higgs-Kibble ϕ-field, while solid ones indicate a W-field.

$$\begin{aligned} V^{\mu\nu} = g^5\, s^4_\theta\, \frac{M_{\scriptscriptstyle W}}{2}\, \mu^{2\epsilon} \int d^n q_1\, d^n q_2 \Bigl\{ \Bigl[(q_2+q_1)\cdot(p_2-p_1) \,-\, q_2^2 \,-\, q_1\cdot q_2\Bigr]\, \delta^{\mu\nu} \\ + 2\Bigl[q_1^\mu\, q_2^\nu + (q_1+q_2)^\mu\, p_1^\nu - (q_1+q_2)^\nu\, p_2^\mu\Bigr] + q_2^\mu\,(q_2-q_1)^\nu + (q_1+q_2)^\mu\, p_2^\nu - (q_1+q_2)^\nu\, p_1^\mu \Bigr\} \\ \times \Bigl[(q_1^2+M^2_{\scriptscriptstyle W})\,(q_1-q_2)^2\,(q_2^2+M^2_{\scriptscriptstyle W})\,((q_2+p_1)^2+M^2_{\scriptscriptstyle W})\,((q_2+P)^2+M^2_{\scriptscriptstyle W})\Bigr]^{-1}, \end{aligned} \tag{44}$$

where $s_\theta\,(c_\theta)$ is the sine (cosine) of the weak-mixing angle. Terms containing q_2^2, $q_1\cdot q_2$ and $q_2\cdot p_1$ are immediately eliminated from the final answer. Consider now terms with q_2^μ (for those with q_1^μ there is an analogous argument); with straightforward substitutions we obtain

$$\int F\,q_2^\mu \quad\to\quad F_1\,p_1^\mu + F_2\,p_2^\mu \quad\to\quad \mathcal{F}_1\,r_1^\mu + \mathcal{F}_2\,r_2^\mu, \tag{45}$$

where with F, etc we indicate some combination of form factors of the V^{I} family whose explicit expression is not relevant for our discussion at this stage. When we project with $P_{D}^{\mu\nu}$ or with $R_{i\nu}$ it follows that

$$\begin{aligned}\int F\,P_{D}^{\mu\nu}\,q_{2\mu} \quad&\to\quad \sum_i \mathcal{F}_i\,P_{D}^{\mu\nu}\,r_{i\mu}=0,\\ \int F\,R_i^\mu\,q_{2\mu} \quad&\to\quad \sum_j \mathcal{F}_j\,R_i\cdot r_j \quad\to\quad \mathcal{F}_i.\end{aligned} \tag{46}$$

When we have a term with $q_2^\mu\,q_2^\nu$ and project with $P_{D}^{\mu\nu}$ it follows

$$\int F\,P_{D}^{\mu\nu}\,q_2^\mu\,q_2^\nu \quad\to\quad \Big[\sum_{ij}\mathcal{F}_{ij}\,r_i^\mu\,r_j^\nu + \mathcal{F}_d\,\delta^{\mu\nu}\Big]\,P_{D,\mu\nu} \;\to\; \mathcal{F}_d\,P_{D,\mu\mu}=\mathcal{F}_d. \tag{47}$$

The number of form factors may be further reduced by requiring that on-(off-)shell WST identities hold.

The procedure that we just illustrated can be easily generalized to other situations; consider, for instance, the off-shell vertex corresponding to $V_1\to V_2+V_3$ where the V_i are gauge bosons. By off-shell we mean that the sources J_{V}^μ emitting/absorbing the vector bosons are not physical (therefore $\partial_\mu\,J_{V}^\mu=0$ is not assumed) and are not on their mass-shell; this choice is also needed when two of the particles correspond to (idealized) stable, physical, vector bosons and we want to check a WST identity. In full generality we write the following decomposition of the vertex:

$$V^{\mu\alpha\beta}=\sum_{i=1}^{2}\Big[A_i\,\delta^{\mu\alpha}\,r_i^\beta + B_i\,\delta^{\mu\beta}\,r_i^\alpha + C_i\,\delta^{\alpha\beta}\,r_i^\mu\Big] + \sum_{ijk=1}^{2} D_{ijk}\,r_i^\mu\,r_j^\alpha\,r_k^\beta. \tag{48}$$

Using the relations

$$R_i\cdot r_j=\frac{1}{2}\delta_{ij},\qquad R_i\cdot R_j=\frac{1}{2}\,G^{-1}_{ij},\qquad \mathcal{G}^{\mu\nu}=R^\mu\,G\,R^\nu=r^\mu\,G^{-1}\,r^\nu, \tag{49}$$

we introduce the following projectors:

$$\begin{aligned}&\mathcal{P}_{Al}^{\mu\alpha\beta}=\delta^{\mu\alpha}\,R_l^\beta-2\,\mathcal{G}^{\mu\alpha}\,R_l^\beta,\quad \mathcal{P}_{Bl}^{\mu\alpha\beta}=\delta^{\mu\beta}\,R_l^\alpha-2\,\mathcal{G}^{\mu\beta}\,R_l^\alpha,\quad \mathcal{P}_{Cl}^{\mu\alpha\beta}=\delta^{\alpha\beta}\,R_l^\mu-2\,\mathcal{G}^{\alpha\beta}\,R_l^\mu,\\ &\mathcal{P}_{Dijl}^{\mu\alpha\beta}=R_i^\mu\,R_j^\alpha\,R_l^\beta-\frac{1}{2(n-2)}\Big[G^{-1}_{ij}\,\mathcal{P}_{Al}^{\mu\alpha\beta}+G^{-1}_{il}\,\mathcal{P}_{Bj}^{\mu\alpha\beta}+G^{-1}_{jl}\,\mathcal{P}_{Ci}^{\mu\alpha\beta}\Big].\end{aligned} \tag{50}$$

Their action can be represented as

$$A_i=\frac{2}{n-2}\,\mathcal{P}_{Ai}^{\mu\alpha\beta}\,V_{\mu\alpha\beta},\quad B_i=\frac{2}{n-2}\,\mathcal{P}_{Bi}^{\mu\alpha\beta}\,V_{\mu\alpha\beta},\quad C_i=\frac{2}{n-2}\,\mathcal{P}_{Ci}^{\mu\alpha\beta}\,V_{\mu\alpha\beta},\quad D_{ijl}=8\,\mathcal{P}_{Dijl}^{\mu\alpha\beta}\,V_{\mu\alpha\beta}. \tag{51}$$

At the level of triple vector boson couplings we encounter an additional complication, namely CP-odd form factors are absent only in the total amplitude but not in single diagrams. Therefore, one should write a more general form for the vertex, including CP-odd terms:

$$V_\epsilon^{\mu\alpha\beta}=\sum_{i=1}^{2}\Big[E_i\,\epsilon(\lambda,\sigma,\mu,\alpha)\,r_i^\beta+F_i\,\epsilon(\lambda,\sigma,\mu,\beta)\,r_i^\alpha+G_i\,\epsilon(\lambda,\sigma,\alpha,\beta)\,r_i^\mu\Big]\,r_{1\lambda}\,r_{2\sigma}. \tag{52}$$

The following property holds: $\mathcal{P}_{Ii}^{\mu\alpha\beta}\,V_{\epsilon,\mu\alpha\beta}=0$, for $I=A,B,C$ and $\mathcal{P}_{Dijk}^{\mu\alpha\beta}\,V_{\epsilon,\mu\alpha\beta}=0$.

For external Proca fields (and also for Rarita-Schwinger fields), however, our preference will be for other methods [23] where the wave-functions for vector particles can be entirely expressed in terms of Dirac spinors with arbitrary polarization vectors allowing for the implementation of projector techniques for helicity amplitudes [20].

5 Techniques for the reduction of two-loop two-point functions

It is well-known that the reduction of two-loop tensor integrals can be achieved up to two-point functions if we are ready to enlarge the class of scalar functions. The original derivation is due to Weiglein, Scharf and Bohm [7]; for completeness we review here the necessary technology and refer the reader to Appendix B for the full list of results.

Standard reduction to scalar integrals amounts to writing down the most general decomposition of tensor integrals, and to transform this relation into a system of linear equations whose unknowns are the form factors and the known terms follow from algebraic reduction between saturated numerators and denominators.

The well-known obstacle on the road to scalarization of multi-loop diagrams is represented by the occurrence of irreducible numerators, i.e. those cases where a $q_i \cdot p$ term appears in the numerator, but no parameterization of the diagram can be found where the inverse propagators $q_i^2 + m_j^2$ and $(q_i + p)^2 + m_l^2$ simultaneously occur. For any two-loop self-energy diagram with I propagators there are $5 - I$ irreducible scalar products. To illustrate the procedure we start considering some simple example, e.g. a vector integral

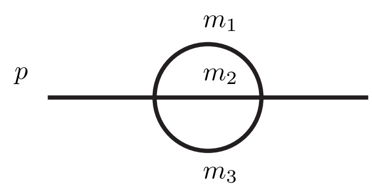


Figure 2: Scalar diagram of the S^A-family, the so-called sunset (or sunrise) configuration.

of the S^A-family, depicted in Fig. 2,

$$S^{A}(\mu \,|\, 0\,;\, p, \{m\}_{123}) = \frac{\mu^{2\epsilon}}{\pi^4} \int d^n q_1 \, d^n q_2 \, \frac{q_{1\mu}}{[1]\,[2]_{A}\,[3]_{A}}, \tag{53}$$

where we introduced a shorthand notation for the inverse propagators:

$$[1] = q_1^2 + m_1^2, \quad [2]_{A} = (q_1 - q_2 + p)^2 + m_2^2, \quad [3]_{A} = q_2^2 + m_3^2. \tag{54}$$

Apparently we meet an irreducible numerator, but we can generalize the procedure considering an intermediate reduction with respect to sub-loops, a technique originally introduced in [7]. In the following subsection we briefly illustrate this technique.

5.1 Reduction to sub-loops

Consider Eq.(53): we may write

$$\int d^n q_1 \, \frac{q_{1\mu}}{[1][2]_{A}} = X_{A}\,(q_2 - p)_{\mu}. \tag{55}$$

If we multiply both sides of Eq.(55) by $(q_2 - p)_\mu$ and use the identities

$$\frac{q_1 \cdot p}{[2]_{A}} = \frac{q_1 \cdot q_2}{[2]_{A}} + \frac{1}{2}\Big[1 - \frac{q_1^2 + q_2^2 - 2\,q_2 \cdot p + m_2^2 + p^2}{[2]_{A}}\Big], \qquad \frac{q_1^2}{[1]} = 1 - \frac{m_1^2}{[1]}, \tag{56}$$

we can solve for X_A obtaining

$$X_{A} = -\frac{1}{2}\,\frac{1}{[0]_{A}} \int d^n q_1 \Big[\frac{m_{12}^2 - [0]_{A}}{[1][2]_{A}} + \frac{1}{[1]} - \frac{1}{[2]_{A}}\Big], \tag{57}$$

where a new propagator has made its appearance, $[0]_{A} = (q_2 - p)^2$. We then use a second pair of identities,

$$\frac{q_2 \cdot p}{[0]_{A}} = -\frac{1}{2}\Big[1 - \frac{q_2^2 + p^2}{[0]_{A}}\Big], \qquad \frac{q_2^2}{[3]_{A}} = 1 - \frac{m_3^2}{[3]_{A}}, \tag{58}$$

to obtain the following result:

$$S^{A}(\mu\,|\,0\,;\,p,\{m\}_{123}) = \frac{1}{2}\,S^{A}(0\,|\,\mu\,;\,p,\{m\}_{123}) + \frac{1}{4}\,S^{A}_{a}\,p_{\mu}, \tag{59}$$

$$S^{A}_{a} = m^2_{12}\,(\frac{m^2_3}{p^2}+1)\,S^{C}_{0}(p,\{m\}_{12},0,m_3) + (\frac{m^2_{12}}{p^2}-2)\,S^{A}_{0}(p,\{m\}_{123})$$
$$-\,\frac{m^2_{12}}{p^2}\,S^{A}_{0}(0,0,\{m\}_{12}) - A_0([m_1,m_2])\,\Big[(1+\frac{m^2_3}{p^2})\,B_0(p,m_3,0) + \frac{1}{p^2}\,A_0(m_3)\Big]. \tag{60}$$

In Eq.(60) we used generalized one-loop scalar functions defined in Eq.(16) and

$$S^{C}_{0}(p,\{m\}_{1234}) = \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\,d^nq_2\,\frac{1}{[1]\,[2]_C\,[3]_C\,[4]_C}, \tag{61}$$

where the propagators are $[2]_C = (q_1-q_2)^2+m^2_2$, $[3]_C = q^2_2+m^2_3$ and $[4]_C = (q_2+p)^2+m^2_4$. Henceforth we continue our derivation for $S^{A}(0\,|\,\mu\,;\,p,\{m\}_{123})$ and write another equation,

$$\int d^nq_2\,\frac{q_{2\mu}}{[2]_A[3]_A} = Y_A\,(q_1+p)_\mu; \tag{62}$$

a solution for Y_A is obtained,

$$Y_A = -\,\frac{1}{2}\,\frac{1}{[0]_{AA}}\int d^nq_2\,\Big[\frac{m^2_{32}-[0]_{AA}}{[2]_A[3]_A} - \frac{1}{[2]_A} + \frac{1}{[3]_A}\Big], \tag{63}$$

with a new propagator defined by $[0]_{AA} = (q_1+p)^2$. It follows that

$$S^{A}(0\,|\,\mu\,;\,p,\{m\}_{123}) = \frac{1}{2}\,S^{A}(\mu\,|\,0\,;\,p,\{m\}_{123}) + \frac{1}{4}\,S^{A}_{b}\,p_{\mu}, \tag{64}$$

$$S^{A}_{b} = -m^2_{32}\,(\frac{m^2_1}{p^2}+1)\,S^{C}_{0}(p,\{m\}_{32},0,m_1) - (\frac{m^2_{32}}{p^2}-2)\,S^{A}_{0}(p,\{m\}_{123})$$
$$+\,\frac{m^2_{32}}{p^2}\,S^{A}_{0}(0,0,\{m\}_{32}) + A_0([m_3,m_2])\,\Big[(1+\frac{m^2_1}{p^2})\,B_0(p,m_1,0) + \frac{1}{p^2}\,A_0(m_1)\Big]. \tag{65}$$

Therefore, using the system of Eqs.(60)–(65) we can solve for both vector integrals in terms of scalar functions.

The full list of results will be given in Section 7. Already from this simple example we see the appearance of generalized scalar loop integrals in the reduction of tensor ones. In the next Section we present the strategy for their evaluation and discuss the general case based on a special set of identities.

6 Integration by parts identities and generalized recurrence relations

A popular and quite successful tool in dealing with multi-loop diagrams, in particular those containing powers of irreducible scalar products, is represented by the integration-by-parts identities (hereafter IBPI) [11]. It is well-known that arbitrary integrals can be reduced [24] to an handful of Master Integrals (MI) using IBPI [11] and Lorentz-invariance identities [26].

For one-loop diagrams IBPI can be written as

$$\int d^nq\,\frac{\partial}{\partial q_\mu}\,\Big[v_\mu\,F(q,p,m_1\cdots)\Big] = 0, \tag{66}$$

where $v = q, p_1 \cdots, p_E$, and E is the number of independent external momenta. By careful examination of the IBPI one can show that all one-loop diagrams can be reduced to a limited set of MI. Here we would like to point out one drawback of this solution. Consider, for instance, the following identity [25],

$$B_0(1,2\,;\,p,m_1,m_2) = \frac{1}{\lambda(-p^2,m_1^2,m_2^2)}\Big\{(n-3)\,(m_1^2-m_2^2-p^2)\,B_0(p,m_1,m_2) + (n-2)\Big[A_0(m_1) - \frac{p^2+m_1^2+m_2^2}{2\,m_2^2}\,A_0(m_2)\Big]\Big\}, \tag{67}$$

where $\lambda(x,y,z)$ is the familiar Källen lambda function $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$. The factor in front of the curly bracket is exactly the BT-factor associated with the diagram; from the general analysis of [3] we know that at the normal threshold the leading behavior of $B_0(1,2)$ is $\lambda^{-1/2}$, so that the reduction to MI apparently overestimates the singular behavior; of course, by carefully examining the curly bracket in Eq.(67) one can derive the right expansion at threshold, but the result, as it stands, is again a source of cancellations/instabilities. Our experience, e.g. with one-loop multi-leg diagrams [3], shows that numerical evaluation following smoothness algorithms (e.g. Bernstein-Tkachov functional relations [17]) does not increase the degree of divergence when going from scalar to tensor integrals.

The IBPI for two-loop diagrams can be written as

$$\int d^nq_1\,d^nq_2\,\frac{\partial}{\partial a_\mu}\Big[b^\mu\,F(q_1,q_2,\{p\},m_1\cdots)\Big] = 0, \qquad a = q_i, \quad b = q_i, p_1\cdots, p_E, \tag{68}$$

where $i = 1,2$, and E is the number of independent external momenta. Again, using IBPI, arbitrary two-loop integrals can be written in terms of a restricted number of MI. The problem remains in the explicit evaluation of the MI; in the following of this Section we want to show that the solution is purely algebraic and, at the same time, we will discuss the relationship with our approach. Consider again the scalar and the two vector integrals in the S^A-family: for them we have

$$S^A(0\,|\,0\,;\,p,\{m\}_{123}) = S^A_0, \quad S^A(\mu\,|\,0\,;\,p,\{m\}_{123}) = S^A_1\,p_\mu, \quad S^A(0\,|\,\mu\,;\,p,\{m\}_{123}) = S^A_2\,p_\mu. \tag{69}$$

Introducing the notation

$$\int \mathcal{D}S_A = \int_0^1 dx \int_0^1 dy\,\Big[x\,(1-x)\Big]^{-\epsilon/2} y^{\epsilon/2-1}, \tag{70}$$

we derive the parametric representation for the scalar and the two vector integrals:

$$S^A_i = \omega^\epsilon\,\Gamma\,(\epsilon-1)\int \mathcal{D}S_A\,P^A_i(x,y)\,\chi_A^{1-\epsilon}(x,y), \tag{71}$$

where $\Gamma\,(z)$ denotes the Euler gamma function, ω is defined in Eq.(15) and where we introduced the auxiliary polynomials

$$P^A_0 = -1, \qquad P^A_1 = x\,(1-y), \qquad P^A_2 = -\,y. \tag{72}$$

The quadratic form χ_A in Eq.(71) is given by $\chi_A = -p^2\,y^2 + (p^2 - m_3^2 + m_x^2)\,y + m_3^2$, with m_x^2 defined in Eq.(12).

The evaluation of the scalar integral was discussed in [1] and can be easily extended to cover the two remaining cases. This simple example can be fully generalized, thus proving that any smoothness algorithm designed for scalar integrals will also be effective to deal with tensor ones; physical observables can be evaluated without using a reduction procedure. Needless to say, however, that when cancellations are at the basis of the result − for instance when testing the WST identities of the theory − scalarization should be attempted; indeed, in these cases the goodness of the result depends crucially on our capability to express the whole set of graphs in terms of a minimal set of integrals.

One way of deriving this result is purely algebraic: to achieve scalarization, which is equivalent to express irreducible tensor integrals in terms of truly scalar functions, we write down generalized functions

$$S_A^{\alpha_1|\alpha_3|\alpha_2}(n\,;\,p,\{m\}_{123}) = \frac{\mu^{2(4-n)}}{\pi^4}\int d^nq_1\,d^nq_2\,\prod_{i=1}^{3}\,[i]_A^{-\alpha_i}, \tag{73}$$

with $[1]_{A} = [1]$, which are defined for arbitrary space-time dimension n. Subsequently we fix n to be $n = \sum_i \alpha_i + 1 - \epsilon$ and obtain

$$S_{A}^{\alpha_1|\alpha_3|\alpha_2}(n\,;\, p, \{m\}_{123}) = -\,\frac{\Gamma\,(\epsilon - 1)}{\prod_i\,\Gamma\,(\alpha_i)}\,\omega^{3-\sum\,\alpha+\epsilon}\int dC_2\, x^{-\rho_1/2}\,(1-x)^{-\rho_2/2}\,(1-y)^{\rho_3}\,y^{\rho_4/2}\,\chi_{A}^{1-\epsilon}, \quad (74)$$

where ω is defined in Eq.(15) and where we introduced powers

$$\rho_1 = 1 + \alpha_1 - \alpha_2 - \alpha_3 + \epsilon,\;\; \rho_2 = 1 + \alpha_2 - \alpha_1 - \alpha_3 + \epsilon,\;\; \rho_3 = \alpha_3 - 1,\;\; \rho_4 = \alpha_1 + \alpha_2 - \alpha_3 - 3 + \epsilon. \qquad (75)$$

According to the work of Tarasov [27] the content of Eq.(74) can be interpreted by saying that we have a scalar integral in shifted space-time dimension and with non-canonical powers of propagators; equivalently, we may interpret it as an integral in the canonical $4-\epsilon$ dimensions, with all powers in the propagators equal to one but with polynomials of Feynman parameters in the numerator. To formally show this equivalence we write

$$S_{i}^{A} = \sum_{j=1}^{2} \omega^{n_j - 4 + \epsilon}\, k_{ij}\, S_{A}^{\alpha_j\,|\,\beta_j\,|\,\gamma_j}(n_j\,;\, p, \{m\}_{123}), \quad i = 1, 2, \qquad (76)$$

with $n_j = \alpha_j + \beta_j + \gamma_j + 1 - \epsilon$, and fix all coefficients and exponents in order to match Eq.(72). A solution is therefore given by $\alpha_1 = 1, \beta_1 = 2, \gamma_1 = 2$, or by $\alpha_2 = 2, \beta_2 = 1, \gamma_2 = 2$, with coefficients $k_{11} = -1,\; k_{12} = 0$ and $k_{21} = 0,\; k_{22} = 1$.

Note that, starting with two-loop vertices and due to the presence of irreducible scalar products, we should abandon the term “scalarization” in favor of a more general concept, namely the reduction to a minimum number of functions that are needed to classify the problem at hand. One can hence adopt a reduction to scalar integrals in shifted dimensions, followed by a solution of generalized recursion relations [27] (which include the IBPI method as a particular case) reducing the large set of integrals to relatively few MI. Alternatively, we can decide to relate the form factors to truly scalar integrals in the same number of dimensions and belonging to contiguous families, and to integrals with contracted and irreducible numerators for which a numerical solution is available; the quality of this latter numerical solution is as good as the one for the scalar configurations.

The two procedures are algebraically equivalent and preference is, to some extent, a matter of taste, although the power of a procedure can only be justified a posteriori by the goodness of the corresponding result. As a matter of fact, a reduction to master integrals is notoriously difficult to achieve when the problem is characterized by several scales. For completeness we stress that Davydychev [28] and, later on, Bern, Dixon and Kosower [29] gave expressions for one-loop tensor integrals with shifted dimensions; Campbell, Glover and Miller [30] discovered good numerical stability for one-loop integrals in higher dimensions; and a simple formula expressing any N-point integral in terms of integrals in higher dimensions was given by Fleischer, Jegerlehner and Tarasov [31].

An example of reduction of generalized functions with the help of IBP techniques is provided by the well-known result that all generalized scalar sunset diagrams with zero external momentum (i.e. vacuum-bubbles) can be fully reduced. To see this we first introduce

$$S_{0}^{A}(\{\alpha\}_3\,;\, 0, \{m\}_{123}) = \frac{\mu^{2\epsilon}}{\pi^4}\int d^n q_1\, d^n q_2 \prod_{i=1}^{3}\,[i(p=0)]_{A}^{-\alpha_i}, \qquad (77)$$

where $[i]_{A}$ is defined in Eq.(54) but with $p = 0$ and $[1]_{A} = [1]$. IBPI reduce all functions in this class to combinations of $S_{0}^{A}(1,1,2\,;\, 0, \{m\}_{123})$ and products of one-loop integrals. For instance we obtain

$$S_{0}^{A}(\{\alpha\}_3\,;\, 0, \{m\}_{123}) = \frac{1}{m_{132}^2}\,\mathcal{S}_{0}^{A}(\{\alpha\}_3\,;\, 0, \{m\}_{123}), \qquad (78)$$

etc, with

$$\mathcal{S}_{0}^{A}(0, \{m\}_{123}) = \frac{\lambda(m_1^2, m_2^2, m_3^2)}{n-3}\, S_{0}^{A}(1,1,2\,;\, 0, \{m\}_{123}) + \frac{n-2}{n-3}\Big[A_0(m_1)\, A_0(m_2)$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\Big(1 - \frac{m_1^2}{m_3^2} + \frac{m_2^2}{m_3^2}\Big)\, A_0(m_1)\, A_0(m_3) - \frac{1}{2}\Big(1 + \frac{m_1^2}{m_3^2} - \frac{m_2^2}{m_3^2}\Big)\, A_0(m_2)\, A_0(m_3)\Big],\\
\mathcal{S}_0^{A}(2,1,1\,;0,\{m\}_{123}) &= m_{213}^2\, S_0^{A}(1,1,2\,;0,\{m\}_{123}) + \frac{n-2}{2}\Big[\frac{1}{m_3^2}\, A_0(m_2)\, A_0(m_3)\\
&\quad - \frac{1}{m_1^2}\, A_0(m_1)\, A_0(m_2) + \Big(\frac{1}{m_1^2} - \frac{1}{m_3^2}\Big)\, A_0(m_1)\, A_0(m_3)\Big],\\
\mathcal{S}_0^{A}(1,2,1\,;0,\{m\}_{123}) &= m_{123}^2\, S_0^{A}(1,1,2\,;0,\{m\}_{123}) + \frac{n-2}{2}\Big[\Big(\frac{1}{m_2^2} - \frac{1}{m_3^2}\Big)\, A_0(m_2)\, A_0(m_3)\\
&\quad - \frac{1}{m_2^2}\, A_0(m_1)\, A_0(m_2) + \frac{1}{m_3^2}\, A_0(m_1)\, A_0(m_3)\Big],
\end{aligned}
\tag{79}
$$

etc. The number of terms in the reduction tends to increase considerably for higher powers in the propagators of the generalized sunset functions but, as we said, all of them can be expressed through the $(1,1,2)$ sunset integral and products of one-loop A_0-functions.

7 Reduction for tensor two-point functions

In this Section we give a full list of results following the method of Weiglein, Scharf and Bohm [7] as derived in Section 5. Scalar two-loop two-point functions are defined by

$$
\begin{aligned}
S_0^{A}(p,\{m\}_{123}) &= \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\, d^nq_2\, \frac{1}{[1]\,[2]_A\,[3]_A},\\
S_0^{C}(p,\{m\}_{1234}) &= \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\, d^nq_2\, \frac{1}{[1]\,[2]_C\,[3]_C\,[4]_C},\\
S_0^{D}(p,\{m\}_{12345}) &= \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\, d^nq_2\, \frac{1}{[1]\,[2]_D\,[3]_D\,[4]_D\,[5]_D},\\
S_0^{E}(p,\{m\}_{12345}) &= \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\, d^nq_2\, \frac{1}{[1]\,[2]_E\,[3]_E\,[4]_E\,[5]_E},
\end{aligned}
\tag{80}
$$

with propagators

$$
[1] = q_1^2 + m_1^2, \qquad [2]_A = (q_1 - q_2 + p)^2 + m_2^2, \quad [3]_A = q_2^2 + m_3^2, \tag{81}
$$

$$
[2]_C = (q_1 - q_2)^2 + m_2^2, \quad [3]_C = q_2^2 + m_3^2, \quad [4]_C = (q_2 + p)^2 + m_4^2, \tag{82}
$$

$$
[2]_D = (q_1 + p)^2 + m_2^2, \quad [3]_D = (q_1 - q_2)^2 + m_3^2, \quad [4]_D = q_2^2 + m_4^2, \quad [5]_D = (q_2 + p)^2 + m_5^2. \tag{83}
$$

$$
[2]_E = (q_1 - q_2)^2 + m_2^2, \quad [3]_E = q_2^2 + m_3^2, \quad [4]_E = (q_2 + p)^2 + m_4^2, \quad [5]_E = q_2^2 + m_5^2, \tag{84}
$$

Propagators $[i]_E$ should not be confused with those appearing in Eq.(117) which refer to a three-point function. These scalar diagrams were investigated in [1], Eq. (89) and Eqs. (146-147) for $S_0^{A} \equiv S_0^{111}$; in [2], Sect. (5.8) for $S_0^{C} \equiv S_0^{121}$, Sect. (7.3) for $S_0^{D} \equiv S_0^{221}$ and Sect. (7.8) for $S_0^{E} \equiv S_0^{131}$. Furthermore, we define form factors according to

$$
\begin{aligned}
&S^I(\mu\,|\,0) = S_1^I\, p_\mu, \quad S^I(0\,|\,\mu) = S_2^I\, p_\mu, \quad S^I(\mu,\nu\,|\,0) = S_{112}^I\, \delta_{\mu\nu} + S_{111}^I\, p_\mu\, p_\nu,\\
&S^I(\mu\,|\,\nu) = S_{122}^I\, \delta_{\mu\nu} + S_{121}^I\, p_\mu\, p_\nu, \quad S^I(0\,|\,\mu,\nu) = S_{222}^I\, \delta_{\mu\nu} + S_{221}^I\, p_\mu\, p_\nu,
\end{aligned}
\tag{85}
$$

with $I = A, C, D, E$; the irreducible classes for two-loop two-point functions are shown in Fig. 3. Generalized one-loop functions are given in Eqs.(16)–(18): after reduction,(with $\lambda_{ij} = \lambda(-p^2, m_i^2, m_j^2)$) we obtain

$$
A_0(\alpha, m) = \frac{\mu^\epsilon}{i\,\pi^2}\int \frac{d^nq}{(q^2+m^2)^\alpha} = \frac{1}{m^2}\Big[1 - \frac{4-\epsilon}{2\,(\alpha-1)}\Big]\, A_0(\alpha-1, m), \quad \mathrm{Re}\,\alpha > 1, \tag{86}
$$

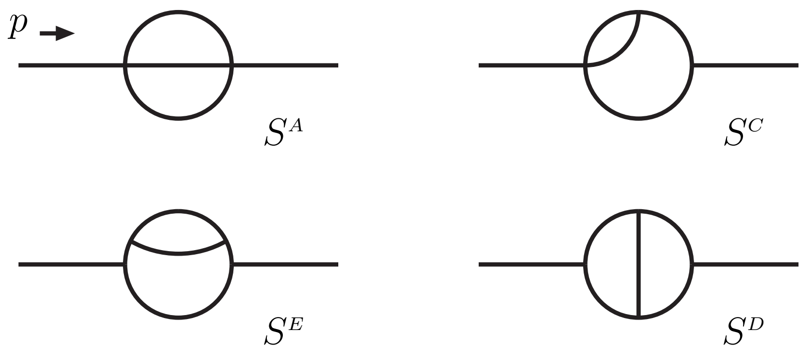


Figure 3: Irreducible classes for two-loop two-point functions.

$$
\begin{aligned}
B_1(2,1\,;\,p,\{m\}_{12}) &= \frac{1}{2\,p^2}\Big[A_0(2,m_1) - B_0(p,\{m\}_{12}) - l_{p12}\,B_0(2,1,p,\{m\}_{12})\Big],\\
B_{21}(2,1\,;\,p,\{m\}_{12}) &= -\frac{1}{4}\,\frac{1}{(n-1)p^4}\,\Big[n\,l_{p12}\,A_0(2,m_1) + n\,A_0([m_1,m_2]) + 2\,(2\,p^2 - n\,l_{p12})\,B_0(p,m_1,m_2)\\
&\quad + (4\,p^2\,m_1^2\,(n-1) - n\,\lambda_{12})\,B_0(2,1\,;\,p,m_1,m_2)\Big],\\
B_{22}(2,1\,;\,p,\{m\}_{12}) &= -\frac{1}{4}\,\frac{1}{(n-1)p^2}\Big[A_0([m_1,m_2]) + l_{p12}\,A_0(2,m_1) + 2\,l_{p21}\,B_0(p,\{m\}_{12})\\
&\quad + \lambda_{12}\,B_0(2,1\,;\,p,\{m\}_{12})\Big],\\
B_1(1,2;\,p,\{m\}_{12}) &= B_1(2,1\,;\,-p,\{m\}_{21}) - B_0(2,1\,;\,-p,\{m\}_{21}),\quad \text{etc.}
\end{aligned}
\tag{87}
$$

In this Section it is always understood that the space-time dimension is $n = 4-\epsilon$. Whenever reducibility is at hand we apply standard methods and obtain the following list of results:

$$
\begin{aligned}
S^C_{111} &= -\frac{1}{p^2}\,\Big[A_0(m_2)\,B_0(p,\{m\}_{34}) + n\,S^C_{112}(p,\{m\}_{1234}) + m_1^2\,S^C_0(p,\{m\}_{1234})\Big],\\
S^C_{121} &= \frac{1}{2}\,\frac{1}{(n-1)\,p^4}\,\Big\{-\,p^2\,A_0([m_1,m_2])\,B_0(p,\{m\}_{34}) + p^2\,m^2_{123}\,S^C_0(p,\{m\}_{1234})\\
&\quad - p^2\,S^A_0(p,\{m\}_{1234}) - n\,\Big[l_{P34}\,S^C_1(p,\{m\}_{1234}) + S^A_1(p,\{m\}_{124})\Big]\Big\}\,,
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
S^C_{221} &= \frac{1}{4}\,\frac{1}{(n-1)\,p^4}\,\Big\{\lambda_{34}\,S^C_0(p,\{m\}_{1234}) - l_{P43}\,\Big[S^A_0(p,\{m\}_{124}) + S^A_0(0,\{m\}_{123})\Big] - 2\,p^2\,S^A_2(p,\{m\}_{124})\\
&\quad + (n-1)\,\Big[(\lambda_{34} - 4\,p^2\,m_3^2)\,S^C(p,\{m\}_{1234}) + (3\,p^2 - m_3^2 + m_4^2)\,S^A_0(p,\{m\}_{124})\\
&\quad - l_{P34}\,S^A_0(0,\{m\}_{123}) - 2\,p^2\,S^A_2(p,\{m\}_{124})\Big]\Big\}\,,
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
S^C_{122} &= \frac{1}{2}\,\frac{1}{n-1}\,\Big[A_0([m_1,m_2])\,B_0(p,\{m\}_{34}) - m^2_{132}\,S^C_0(p,\{m\}_{123}) + S^A_0(p,\{m\}_{124})\\
&\quad + l_{P34}\,S^C_1(p,\{m\}_{1234}) + S^A_1(p,\{m\}_{124})\Big]\,,
\end{aligned}
\tag{90}
$$

$$
S^C_{222} = \frac{1}{4}\frac{1}{(n-1)p^2}\Big\{-\lambda_{34}S^C_0(p,\{m\}_{1234}) + l_{P43}\,\Big[S^A_0(p,\{m\}_{124}) + S^A_0(0,\{m\}_{123})\Big] + p^2 S^A_2(p,\{m\}_{124})\Big\},
\tag{91}
$$

$$S_2^C = \frac{1}{2\,p^2}\left[-l_{P34}\,S_0^C(p,\{m\}_{1234}) - S_0^A(p,\{m\}_{124}) + S_0^A(0,\{m\}_{124})\right], \tag{92}$$

$$S_{111}^E = -\frac{1}{p^2}\left[A_0(m_2)\,B_0(2,1\,;\,p,\{m\}_{34}) + n\,S_{112}^E(p,\{m\}_{12343}) + m_1^2\,S_0^E(p,\{m\}_{12343})\right], \tag{93}$$

$$\begin{aligned} S_{121}^E = \frac{1}{2}\,\frac{1}{(n-1)p^2}\Big[&A_0([m_1,m_2])\,C_0(p,-p,\{m\}_{343}) + m_{123}^2\,S_0^E(p,\{m\}_{12343}) - S_0^C(p,\{m\}_{1234}) \\ &- n\,l_{P34}\,S_1^E(p,\{m\}_{12343}) - n\,S_1^C(p,\{m\}_{1234})\Big], \end{aligned} \tag{94}$$

$$\begin{aligned} S_{221}^E = \frac{1}{4}\,\frac{1}{(n-1)p^4}\Big\{&(n\,l_{p34}^2 + 4\,p^2\,m_3^2)\,S_0^E(p,\{m\}_{12343}) + 2\,(n\,l_{p34} - 2\,p^2)\,S_0^C(p,\{m\}_{1234}) \\ &- n\,l_{p34}\,S_0^C(0,\{m\}_{1233}) + n\left[S_0^A(p,\{m\}_{124}) - S_0^A(0,\{m\}_{123})\right]\Big\}, \end{aligned} \tag{95}$$

$$\begin{aligned} S_{122}^E = \frac{1}{4}\,\frac{1}{(n-1)}\Big[&A_0([m_1,m_2])\,B_0(2,1\,;\,p,\{m\}_{34}) - m_{123}^2\,S_0^E(p,\{m\}_{12343}) + S_0^C(p,\{m\}_{1234}) \\ &+ l_{p34}\,S_1^E(p,\{m\}_{12343}) + S_1^C(p,\{m\}_{1234})\Big], \end{aligned} \tag{96}$$

$$\begin{aligned} S_{222}^E = \frac{1}{4}\,\frac{1}{(n-1)p^2}\Big[&-\lambda_{34}\,S_0^E(p,\{m\}_{12343}) + 2\,l_{p43}\,S_0^C(p,\{m\}_{1234}) \\ &+ l_{p34}\,S_0^C(0,\{m\}_{1233}) - S_0^A(p,\{m\}_{124}) + S_0^A(0,\{m\}_{123})\Big], \end{aligned} \tag{97}$$

$$S_2^E = \frac{1}{2\,p^2}\Big[-l_{p34}\,S_0^E(p,\{m\}_{12343}) - S_0^C(p,\{m\}_{1234}) + S_0^C(0,\{m\}_{1233})\Big], \tag{98}$$

$$\begin{aligned} S_{111}^D = \frac{1}{4}\,\frac{1}{(n-1)\,p^4}\Big\{&(n\,l_{p12}^2 + 4\,m_1^2\,p^2)\,S_0^D(p,\{m\}_{12345}) - n\,l_{p12}\,S_0^C(p,\{m\}_{1345}) \\ &- \left[4\,p^2 - n\,(3\,p^2 - m_{12}^2)\right]S_0^C(p,\{m\}_{2354}) + 2\,n\,p^2\left[S_1^C(p,\{m\}_{1345}) + S_1^C(p,\{m\}_{2354})\right]\Big\}, \end{aligned} \tag{99}$$

$$\begin{aligned} S_{121}^D = \frac{1}{4}\,\frac{1}{(n-1)\,p^4}\Big\{&-2\,p^2\,B_0(p,\{m\}_{12})\,B_0(p,\{m\}_{45}) \\ &+ (n\,(p^4 + p^2\,m_{245}^2 - p^2\,m_1^2 + m_{12}^2\,m_{45}^2) + 2\,m_{134}^2\,p^2)\,S_0^D(p,\{m\}_{12345}) \\ &- n\,l_{p45}\,S_0^C(p,\{m\}_{1345}) - (2\,p^2 - n\,l_{p45})\,S_0^C(p,\{m\}_{2354}) - n\,l_{p12}\,S_0^C(p,\{m\}_{4312}) \\ &- (2\,p^2 - n\,l_{p12})\,S_0^C(p,\{m\}_{5321}) - n\left[S_0^A(p,\{m\}_{432}) + S_0^A(p,\{m\}_{531})\right] \\ &+ n\left[S_0^A(0,\{m\}_{431}) + S_0^A(0,\{m\}_{532})\right]\Big\}, \end{aligned} \tag{100}$$

$$\begin{aligned} S_{221}^D = \frac{1}{4}\,\frac{1}{(n-1)\,p^4}\Big\{&(4\,m_4^2\,p^2 + n\,l_{p45}^2)\,S_0^D(p,\{m\}_{12345}) - n\,l_{p54}\,S_0^C(p,\{m\}_{4312}) \\ &- \left[4\,p^2 - n\,(3\,p^2 - m_{45}^2)\right]\,S_0^C(p,\{m\}_{5321}) + 2\,n\,p^2\,S_1^C(p,\{m\}_{4312}) + 2\,n\,S_1^C(p,\{m\}_{5321})\Big\}, \end{aligned} \tag{101}$$

$$S^{D}_{112} = \frac{1}{4}\,\frac{1}{(n-1)\,p^2}\,\Big[-\lambda_{12}\,S^{D}_{0}(p,\{m\}_{12345}) + l_{p12}\,S^{C}_{0}(p,\{m\}_{1345}) + l_{p21}\,S^{C}_{0}(p,\{m\}_{2354})$$
$$-\,2\,p^2\,S^{C}_{1}(p,\{m\}_{1345}) + 2\,p^2 S^{C}_{1}(p,\{m\}_{2354})\Big]\,, \qquad (102)$$

$$S^{D}_{122} = \frac{1}{4}\,\frac{1}{(n-1)\,p^2}\,\Big[2\,p^2\,B_0(p,\{m\}_{12})\,B_0(p,\{m\}_{45})$$
$$+\,\Big[m^2_{12}\,m^2_{54} - p^2\,(m^2_1 + m^2_2 - 2m^2_3 + m^2_4 + m^2_5) - p^4\Big]\,S^{D}_{0}(p,\{m\}_{12345})$$
$$+\,l_{p45}\,S^{C}_{0}(p,\{m\}_{1345}) + l_{p54}\,S^{C}_{0}(p,\{m\}_{2354}) + l_{p12}\,S^{C}_{0}(p,\{m\}_{4312}) + l_{p21}\,S^{C}_{0}(p,\{m\}_{5321})$$
$$+\,S^{A}_{0}(p,\{m\}_{432}) + S^{A}_{0}(p,\{m\}_{531}) - S^{A}_{0}(0,\{m\}_{431}) - S^{A}_{0}(0,\{m\}_{532})\Big]\,, \qquad (103)$$

$$S^{D}_{222} = \frac{1}{4}\,\frac{1}{(n-1)\,p^2}\,\Big[-\lambda_{45}\,S^{D}_{0}(p,\{m\}_{12345}) + l_{p45}\,S^{C}_{0}(p,\{m\}_{4312}) + l_{p54}\,S^{C}_{0}(p,\{m\}_{5321})$$
$$-\,2\,p^2\,S^{C}_{1}(p,\{m\}_{4312}) - 2\,p^2\,S^{C}_{1}(p,\{m\}_{5321})\Big]\,, \qquad (104)$$

$$S^{D}_{1} = \frac{1}{2p^2}\,\Big[-\,l_{p12}\,S^{D}_{0}(p,\{m\}_{12345}) + S^{C}_{0}(p,\{m\}_{1345}) - S^{C}_{0}(p,\{m\}_{2354})\Big]\,, \qquad (105)$$

$$S^{D}_{2} = \frac{1}{2p^2}\,\Big[-\,l_{p45}\,S^{D}_{0}(p,\{m\}_{12345}) + S^{C}_{0}(p,\{m\}_{4312}) - S^{C}_{0}(p,\{m\}_{5321})\Big]\,. \qquad (106)$$

The standard reduction procedure does not work for the S^{C}_{1} and S^{E}_{1} form factors, since for them the scalar product $p\cdot q_1$ is irreducible. In order to express these form factors in term of other scalar functions, it is then necessary to employ the procedure outlined in Section 5, i.e. considering first the scalarization with respect to the sub-loops. Employing Eq.(131) one obtains the following relations

$$S^{C}_{1} = \frac{1}{4}\,\frac{1}{m^2_3 p^2}\Big\{A_0([m_1,m_2])\,\Big[A_0(m_3) - B_0(p,\{m\}_{34})\,l_{p34} + B_0(p,0,m_4)\,(p^2+m^2_4)\Big]$$
$$-\,S^{C}_{0}(p,\{m\}_{1234})\,l_{p34}\,m^2_{123} + S^{C}_{0}(p,\{m\}_{12},0,m_4)\,m^2_{12}\,(p^2+m^2_4) - S^{A}_{0}(p,\{m\}_{124})\,m^2_3$$
$$+\,S^{A}_{0}(0,\{m\}_{123})\,m^2_{123} - S^{A}_{0}(0,\{m\}_{12},0)\,m^2_{12}\Big\}\,, \qquad (107)$$

$$S^{E}_{1} = \frac{1}{4}\,\frac{1}{m^4_3 p^2}\Big\{A_0([m_1,m_2])\,\Big[\frac{4-n}{2}\;A_0(m_3) - B_0(p,\{m\}_{34})\,(p^2+m^2_4) + B_0(p^2,0,m_4)\,(p^2+m^2_4)$$
$$-\,m^2_3\,l_{p34}\,B_0(2,1\,;\,p,\{m\}_{34})\Big] - m^2_3\,l_{p34}\,m^2_{123}\,S^{E}_{0}(p,\{m\}_{12343})$$
$$-\,\Big[(m^2_{12}\,(p^2+m^2_4) + m^4_3\Big]\,S^{C}_{0}(p,\{m\}_{1234})$$
$$+\,m^2_{12}\,(p^2+m^2_4)\,S^{C}_{0}(p,\{m\}_{12},0,m_4)\; + m^2_3\,m^2_{123}\,S^{C}_{0}(0,\{m\}_{1233})$$
$$+\,m^2_{12}\,\Big[S^{A}_{0}(0,\{m\}_{123}) - S^{A}_{0}(0,\{m\}_{12},0)\Big]\Big\}\,. \qquad (108)$$

8 Strategies for the evaluation of two-loop self-energies

In this Section we provide an explicit example of possible strategies to evaluate diagrams with a non-trivial spin structure. Consider the diagram in Fig. 4, representing one of the two-loop contributions to the Z-boson self-energy (the diagram may be needed to assemble the components of a scattering amplitude or to

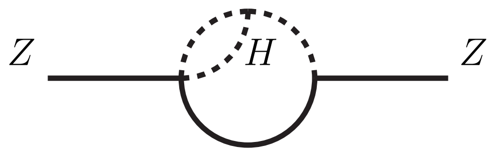


Figure 4: Example of a diagram belonging to the S^C-family and contributing to the Z self-energy. Dashed lines represent a H-field.

compute a doubly-contracted WST identity, in which case we have to multiply the corresponding expression by $p_\mu p_\nu$). In the R_ξ-gauge, with $[\xi] = (q_2+p)^2 + \xi^2 M_Z^2$ the diagram is be written as

$$
\begin{aligned}
S_{\mu\nu} = & -\frac{3}{8}\frac{g^4 M_H^2}{c_\theta^4}\mu^{2\epsilon}\int d^n q_1\, d^n q_2 \Big\{\frac{\delta_{\mu\nu}}{[\xi=1]} + \frac{(q_2+p)_\mu (q_2+p)_\nu}{M_Z^2}\Big[\frac{1}{[\xi=1]} - \frac{1}{[\xi_Z]}\Big]\Big\} \\
& \times \frac{1}{(q_1^2+M_H^2)\,[(q_1-q_2)^2+M_H^2]\,(q_2^2+M_H^2)} = -\frac{3}{8}\frac{g^4\pi^4 M_H^2}{c_\theta^4 M_Z^2}\Big(\Pi^d\,\delta_{\mu\nu} + \Pi^p\,p_\mu\,p_\nu\Big). \qquad (109)
\end{aligned}
$$

After multiplication by $p_\mu p_\nu$ we can perform all the algebraic manipulations, like rewriting $q_2 \cdot p$ and q_2^2 in terms of propagators, or we can use Eq.(85) and the results of Section 7 in order to obtain a fully scalarized expression. Alternatively, again using Eq.(85), we can write

$$
\begin{aligned}
\Pi^d &= S^C_{222}(1) - S^C_{222}(\xi_Z) + M_Z^2\,S^C_0(1), \\
\Pi^p &= S^C_{221}(1) - S^C_{221}(\xi_Z) + 2\,S^C_2(1) - 2\,S^C_2(\xi_Z) + S^C_0(1) - S^E_0(\xi_Z), \qquad (110)
\end{aligned}
$$

where we explicitly indicated the dependence on the gauge parameter ξ_Z. To derive an explicit expression for the form factors we decompose the diagram according to $S = S_{DP} + S_{SP} + S_F$, where the subscripts refer to double and single ultraviolet poles and to the finite ultraviolet part. Note, however, that the splitting is defined only modulus constants. The three components of the result (with a presentation limited here to the $\xi = 1$ part) are given in the following list:

$$
S^C_{0\,;\,DP} = -\frac{1}{\epsilon^2} - \overline{\Delta}^2_{UV}, \qquad S^C_{2\,;\,DP} = -\frac{1}{2}S^C_{0\,;\,DP}, \qquad S^C_{221\,;\,DP} = \frac{1}{3}S^C_{0\,;\,DP},
$$

$$
S^C_{222\,;\,DP} = (\frac{1}{6}p^2 + \frac{49}{6}M_H^2 + \frac{1}{2}M_Z^2)\,\overline{\Delta}_{UV}\,\frac{1}{\epsilon} + (\frac{1}{12}p^2 - \frac{5}{4}M_H^2 + \frac{1}{4}M_Z^2)\,\Delta^2_{UV}, \qquad (111)
$$

$$
S^C_{0\,;\,SP} = 2\,\overline{\Delta}_{UV}\Big[\int_0^1 dx\,\ln\chi(x) - \frac{1}{2}\Big], \qquad S^C_{2\,;\,SP} = \frac{1}{4}\overline{\Delta}_{UV}\Big[1 - 8\int_0^1 dx\,x\,\ln\chi(x)\Big],
$$

$$
S^C_{221\,;\,SP} = 2\,\overline{\Delta}_{UV}\Big[\int_0^1 dx\,x^2\,\ln\chi(x) - \frac{1}{36}\Big],
$$

$$
\begin{aligned}
S^C_{222\,;\,SP} = & \frac{20}{9}M_H^2\,\Delta_{UV} - (\frac{5}{72}p^2 + \frac{97}{8}M_H^2 + \frac{13}{24}M_Z^2)\,\overline{\Delta}_{UV} + \frac{23}{3}M_H^2\,\ln\mu_H^2\,\overline{\Delta}_{UV} \\
& + \overline{\Delta}_{UV}\int dCS\,(x\,;\,y,z)\,\Big\{-\Big[3\,y\,M_H^2 + 3\,(p^2 - M_H^2 + M_Z^2)\,z - 4\,z^2\,p^2\Big]\ln\chi(x,y,z) \\
& + M_H^2\,(3\,y-2)\Big[\frac{\ln\chi(x,y,z)}{x}\Big|_+ - \frac{\ln\chi(x,y,z)}{x-1}\Big|_+\Big]\Big\}, \qquad (112)
\end{aligned}
$$

$$S^C_{F} = \int dCS\,(x\,;\,y,z)\,\frac{\ln\chi(x,y,z)}{1-y}\Big|_{+} + \int_0^1 dx\,\ln\chi(x)\,\Big[L_1(x)+2\Big] - \frac{3}{2} - \frac{1}{2}\,\zeta(2),$$

$$\begin{aligned}
S^C_{2\,;\,F} &= -\int dCS\,(x\,;\,y,z)\;z\frac{\ln\chi(x,y,z)}{1-y}\Big|_{+} - \int_0^1 dx\,x\,\ln\chi(x)\,\Big[L_1(x)+2\Big] + \frac{11}{16} + \frac{1}{4}\,\zeta(2),\\
S^C_{221\,;\,F} &= \int dCS\,(x\,;\,y,z)\;z^2\,\frac{\ln\chi(x,y,z)}{1-y}\Big|_{+} + \int_0^1 dx\,x^2\,\ln\chi(x)\,\Big[L_1(x)+2\Big] - \frac{97}{216} - \frac{1}{6}\,\zeta(2),\\
S^C_{222\,;\,F} &= \int dCS\,(x\,;\,y,z)\,\Big\{M^2_{H}\,(1-\frac{3}{2}\,y)\ln(1-x)\,\Big(\frac{\ln\chi(x,y,z)}{x}\Big|_{+}\Big)\\
&+ \frac{3}{2}\,\Big[y\,M^2_{H} + (p^2 - M^2_{H} + M^2_{Z})\,z - 6\,z^2\,p^2\Big]\,\ln\chi(x,y,z)\,L_2(x,y,z)\\
&- \frac{5}{2}\,\Big[y\,M^2_{H} + (p^2 - M^2_{H} + M^2_{Z})\,z - z^2\,p^2\Big]\,\ln\chi(x,y,z)\\
&+ M^2_{H}\,(\frac{3}{2}\,y-1)\ln^2(1-x)\,\Big(\frac{\ln\chi(x,y,z)}{x}\Big|_{+}\Big) + \frac{5}{2}\,M^2_{H}\,(1-y)\,\Big[\frac{\ln\chi(x,y,z)}{x-1}\Big|_{+} - \frac{\ln\chi(x,y,z)}{x}\Big|_{+}\Big]\\
&+ M^2_{H}\,(\frac{3}{2}\,y-1)\,\Big[\frac{\ln\chi(x,y,z)\,L_2(x,y,z)}{x-1}\Big|_{+} - \frac{\ln\chi(x,y,z)\,L_2(x,y,z)}{x}\Big|_{+}\Big]\Big\} + \frac{5}{36}\,M^2_{H}\,\ln\mu^2_h\\
&+ \Big[\frac{1}{24}\,p^2 - \frac{43}{24}\,M^2_{H} + \frac{1}{8}\,M^2_{Z}\Big]\,\zeta(2) + \frac{145}{864}\,p^2 - \frac{1}{3}\,\zeta(3)\,M^2_{H} + \frac{9247}{864}\,M^2_{H} + \frac{251}{288}\,M^2_{Z}. \qquad (113)
\end{aligned}$$

Here $\zeta(n)$ denotes the Riemann zeta function. To derive our result we introduced the auxiliary functions

$$\begin{aligned}
\chi(x) &= x\,(1-x)\,s_p + \mu^2_{H}\,(1-x) + \mu^2_{Z}\,x, \quad \mu^2_x = \mu^2_{H}\,\Big(\frac{1}{x} + \frac{1}{1-x}\Big),\\
\chi(x,y,z) &= x\,(1-x)\,\Big[z\,(1-z)\,s_p + \mu^2_{H}\,(y-z) + \mu^2_{Z}\,z\Big] + \mu^2_x\,(1-y), \qquad (114)
\end{aligned}$$

where $s_p = \mathrm{sign}\,(p^2)$ and $\mu^2_{H} = M^2_{H}/\mid p^2\mid$, $\mu^2_{Z} = M^2_{Z}/\mid p^2\mid$. We define $L_{1,2}$,

$$L_1(x) = \ln(1-x) - \ln\chi(x)\,, \qquad L_2(x,y,z) = \ln\chi(x,y,z) - \ln(1-y) - \ln x - \ln(1-x). \qquad (115)$$

The $'+'$-distribution is used according to the definitions of Eq.(7).

In conclusion we may say that Eqs.(111)–(113) prove that whenever we can find an algorithm of smoothness for scalar integrals, then the same algorithm can be generalized to handle tensor integrals. The whole diagram, or even sets of diagrams, can be successively mapped into one multi-dimensional integral. The only additional complication is represented by those cases where the scalar diagram is ultraviolet convergent while tensors of the same family diverge; in this case one cannot set $\epsilon = 0$ from the very beginning but, apart from this caveat, the procedure will be essentially the same.

9 Reduction of tensor two-loop three-point functions

In this Section we move to the complex environment of three-point functions. Two independent external momenta induce seven scalar products containing q_1 and/or q_2, and the number of irreducible ones is $7-I$ where I is the number of internal lines in the diagram (note that $4 \le I \le 6$); the choice of the set of irreducible scalar products has, of course, some arbitrariness. In any case, for two-loop diagrams we never have complete reducibility with respect to both q_1 and q_2. Actually, in evaluating observables for physical processes, we encounter a more general situation: massive SM gauge bosons are unstable particles and final states are always made up of stable fermions and/or photons. Referring to Fig. 5 we have propagators that depend on p_1 and p_2+p_3, therefore losing full reducibility in the q_2 sub-loop. The whole procedure will be developed on a diagram-by-diagram basis with the double goal of writing explicit integral representations for all form factors and of deriving a suitable algorithm to express them in terms of ordinary and generalized

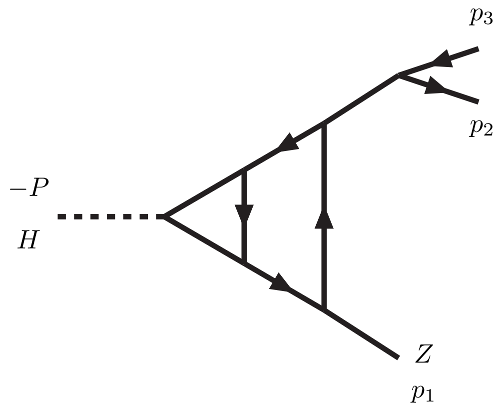


Figure 5: A contribution of the V^K family to $H \to Z^* Z \to Z\overline{f}f$. External momenta flow inwards.

scalar functions. Therefore, for each set of graphs, we will show that all integrals can be expressed in terms of generalized scalar functions, part of which should be subsequently treated within the context of generalized recurrence relations [27]; the final answer will contain a limited number of master integrals.

Alternatively, and this represents our preferred solution, all the integrals not belonging to $\mathcal{S}_4$ – the class of ordinary scalar functions in $n = 4 - \epsilon$ dimensions – can be evaluated according to the given integral representation, following the same lines that we have already adopted in II and in III for solving the problem in the $\mathcal{S}_4$ class.

For each diagram there are many equivalent ways to assign loop momenta; we will make a specific choice for the matrix η of Eq.(2) (the defining parametric representation of the graph) and stick to it also when diagrams of a given family appear in the result of the reduction of the tensor integrals of other families. In these cases the necessary permutations of momenta should be performed, as it will be shown in Section 10.

In our presentation the different families are ordered according to the choice made in [5], where the scalar members were computed explicitly and where the ordering was dictated by a criterion of increasing complexity in the evaluation and by the fact that three graphs belong to the same class V^{1N1}. Therefore, in the following subsections we present and discuss techniques for treating $\{V^E, V^I, V^M\} \in V^{1N1}$ and the more complicated ones, V^G, V^K and V^H. A complete summary of all results for reduction of three-point functions is provided in Appendix B.

9.1 The V^E-family ($\alpha = 1, \beta = 2, \gamma = 1$)

We start our analysis considering the scalar member of the V^E-family of Fig. 6, which is representable as

$$\pi^4\, V_0^E(p_2, P, \{m\}_{1234}) = \mu^{2\epsilon} \int d^n q_1 \int d^n q_2\, \frac{1}{[1][2]_E[3]_E[4]_E}, \tag{116}$$

with propagators defined by

$$[1] \equiv q_1^2 + m_1^2, \quad [2]_E \equiv (q_1 - q_2)^2 + m_2^2, \quad [3]_E \equiv (q_2 + p_2)^2 + m_3^2, \quad [4]_E \equiv (q_2 + P)^2 + m_4^2. \tag{117}$$

Note the symmetry property $V_0^E(p_2, P, \{m\}_{1234}) = V_0^E(P, p_2, \{m\}_{1243})$, besides the one shown in Eq.(438) of Appendix C.

The scalar diagram is overall divergent and so is the (α, γ) sub-diagram. Vector and tensor integrals for all classes usually show additional ultraviolet divergences which have been transferred from the momentum integration to the parametric one. Also for this reason we will keep the n dependence explicit, i.e. $n \neq 4\,(\epsilon \neq 0)$ in all parametrizations. In the following we will discuss vector and rank two tensor integrals for all families. Rank three tensor are fully analyzed in Section 9.7.

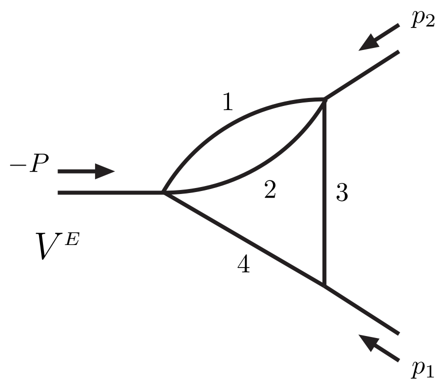


Figure 6: The irreducible two-loop vertex diagrams V^E. External momenta flow inwards. Internal masses are enumerated according to the parametrization of Eq.(117).

9.1.1 Vector integrals in the V^E family

We also consider the V^E vector integrals and introduce the following decomposition in terms of the p_1 , p_2 basis:

$$V^E(\mu\,|\,0\,;\,p_2,P,\{m\}_{1234}) = \sum_{i=1,2} V^E_{1i}(p_2,P,m_{1234})\,p_{i\mu},$$
$$V^E(0\,|\,\mu\,;\,p_2,P,\{m\}_{1234}) = \sum_{i=1,2} V^E_{2i}(p_2,P,\{m\}_{1234})\,p_{i\mu}. \tag{118}$$

Note that we always use the convention $V^X(p\,|\,0\,;\cdots) = p^\mu\,V^X(\mu\,|\,0\,;\cdots)$.

V^E will often appear in the reduction of the form factors belonging to other families and special care should be applied in writing the correct list of arguments. To help understanding this list we rewrite V^E_{ij} according to the following equation:

$$\frac{\mu^{2\epsilon}}{\pi^4}\int d^nr_1d^nr_2\,r_{i\mu}\prod_{l=a}^{d} D_l^{-1} \equiv V^E_{i1}(k_c,k_d,\{m\}_{abcd})\,(k_d-k_c)_\mu\,+V^E_{i2}(k_c,k_d,\{m\}_{abcd})\,k_{c\mu}, \tag{119}$$

where the propagators are now generically written as

$$D_a = r_1^2+m_a^2\,,\quad D_b=(r_1-r_2)^2+m_b^2\,,\quad D_c=(r_2+k_c)^2+m_c^2\,,\quad D_d=(r_2+k_d)^2+m_d^2. \tag{120}$$

Here $i = 1,2$ and $m_a,\dots,m_d$ are generic masses, k_c and k_d are the external momenta appearing in the propagators D_c and D_d, respectively, and r_1, r_2 are the loop momenta. Note that the following identities hold:

$$V^E_{i1}(c,d) = -V^E_{i1}(d,c)+V^E_{i2}(d,c),\qquad V^E_{i2}(c,d)=V^E_{i2}(d,c), \tag{121}$$

where $(c,d) = (k_c,k_d,\{m\}_{abcd})$ etc. Therefore, Eqs.(119)–(120) tell us how to identify the proper list of arguments when these integrals appear as the result of a reduction of tensor integrals belonging to other classes and a permutation has been applied in order to conform to the convention of Eq.(120).

As we explained earlier, all these form factors could be computed directly, without having to perform a reduction. For this reason it is important to list their integral representation. The explicit expression for the vector form factors of this family is

$$V^E_{ij} = -\,\Gamma\,(\epsilon)\int \mathcal{D}V_E\,P_{ij;E}\,\chi_E^{-\epsilon}(x,y,z), \tag{122}$$

$$P_{00;E}=1,\quad P_{11;E}=-x\,z,\quad P_{12;E}=-x\,y,\quad P_{21;E}=-z,\quad P_{22;E}=-y, \tag{123}$$

with an integration measure defined as follows:

$$\int \mathcal{D}V_{E} = \omega^{\epsilon} \int dCS\,(x\,;\,y,z)\,\Big[x\,(1-x)\Big]^{-\epsilon/2}\,(1-y)^{\epsilon/2-1}, \tag{124}$$

where ω is defined in Eq.(15) and where, with our choice for the Feynman parameters, the polynomial χ_{E} is given by

$$\chi_{E}(x,y,z) = -\,F(z,y) + (p_2^2 - m_x^2 + m_3^2)\,y + (2\,p_{12} + l_{134})\,z + m_x^2, \tag{125}$$

where we used Eq.(12). All these functions can be manipulated according to the procedure introduced in III and they will give rise to smooth integral representations.

The generic scalar function in this family is

$$\begin{aligned}
V_{E}^{\alpha_1|\alpha_3,\alpha_4|\alpha_2}(n = \sum_{i=1}^{4}\alpha_i - \epsilon) &= \pi^{-4}(\mu^2)^{4-n}\int d^nq_1 \int d^nq_2 \prod_{i=1}^{4}[i]_{E}^{-\alpha_i} \\
= -\frac{\Gamma\,(\epsilon)}{\prod_{i=1}^{4}\Gamma\,(\alpha_i)}\omega^{\rho_0}&\int dCS\,(x\,;\,y,z)\,x^{\rho_1}\,(1-x)^{\rho_2}\,(1-y)^{\rho_3}\,(y-z)^{\rho_4}\,z^{\rho_5}\,\chi_{E}^{-\epsilon}(x,y,z),
\end{aligned} \tag{126}$$

where $[1]_{E} \equiv [1]$, ω is defined in Eq.(15), with powers $\rho_0 = 4 - \sum_{j=1}^{4}\alpha_j + \epsilon$ and

$$\begin{aligned}
\rho_1 &= \frac{1}{2}\,(-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - 1 - \frac{\epsilon}{2}, \qquad \rho_2 = \frac{1}{2}\,(\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4) - 1 - \frac{\epsilon}{2}, \\
\rho_3 &= \frac{1}{2}\,(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) - 1 + \frac{\epsilon}{2}, \qquad \rho_4 = \alpha_3 - 1, \qquad \rho_5 = \alpha_4 - 1.
\end{aligned} \tag{127}$$

Henceforth, for the form factors of Eq.(118) we can write

$$V_{ij}^{E} = \sum_{l=1}^{4}\omega^{n_{lij}-4+\epsilon}\,k_{lij}\,V_{E}^{\alpha_{1lij}\,|\,\alpha_{3lij},\alpha_{4lij}\,|\,\alpha_{2lij}}(n_{lij}), \tag{128}$$

with ω defined in Eq.(15) and $n_{lij} = \sum_{k=1}^{4}\alpha_{klij} - \epsilon$. The coefficients k_{lij} and the exponents α_{lij}, can be easily read out of the following explicit expressions:

$$\begin{aligned}
V_{11}^{E} &= -\omega^2\,V_{E}^{1\,|\,1,2\,|\,2}, \quad V_{12}^{E} = -\omega^2\,\Big[V_{E}^{1\,|\,2,1\,|\,2} + V_{E}^{1\,|\,1,2\,|\,2}\Big], \quad V_{21}^{E} = -\omega^2\,\Big[V_{E}^{2|1,2|1} + V_{E}^{1|1,2|2}\Big], \\
V_{22}^{E} &= -\omega^2\,\Big[V_{E}^{2|2,1|1} + V_{E}^{2|1,2|1} + V_{E}^{1|2,1|2} + V_{E}^{1|1,2|2}\Big],
\end{aligned} \tag{129}$$

all to be evaluated for $n = 6 - \epsilon$. As usual, there still is the problem of evaluating the integrals of Eq.(129) by means of recurrence relations or, in other words, to link all of them to MI and to develop an algorithm to evaluate the master integrals.

Based on our experience with one-loop multi-leg diagrams, we propose an alternative: an algorithm for the evaluation of tensor integrals offering the same stability characteristics as for scalar integrals. More precisely, we mean a result which is, from a numerical point of view, of the same degree of stability for all integrals and where the real nature of any singularity, apparent or not, is independent of the rank of the integral under consideration. Let us start with $V^{E}(\mu\,|\,0)$, where sub-loop reduction techniques may be applied giving

$$\int d^nq_1\,\frac{q_{1\mu}}{[1][2]_{E}} = X_{E}\,q_{2\mu}, \tag{130}$$

and where X_{E} by standard methods is computed to be

$$X_{E} = \frac{1}{2}\int d^nq_1\,\Big\{\frac{1}{[1][2]_{E}} + \frac{1}{[0]_{E}}\,\Big[\frac{m_{21}^2}{[1][2]_{E}} - \frac{1}{[1]} + \frac{1}{[2]_{E}}\Big]\Big\}, \tag{131}$$

with $[0]_E = q_2^2$. As a consequence of this result we obtain

$$V^E(\mu\,|\,0\,;\,p_2, P, \{m\}_{1234}) = \frac{1}{2}\Big[m_{21}^2\, V^I(0\,|\,\mu\,;\,p_2, P, \{m\}_{12}, 0, \{m\}_{34}) + V^E(0\,|\,\mu\,;\,p_2, P, \{m\}_{1234}) - C_\mu(p_2, p_1, 0, \{m\}_{34})\, A_0([m_2, m_1])\Big], \tag{132}$$

so that the $q_{1\mu}$ vector integral in the E-family is related to the $q_{2\mu}$ vector integrals of the $I-E$ families. The function C_μ in Eq.(132) is defined in Eq.(20), the I family will be discussed in Section 9.2. For the V^E-family we have partial reducibility, i.e. $V^E(0|\,p_1)$ can be expressed in term of known quantities:

$$V^E(0|\,p_1; p_2, P, \{m\}_{1234}) = -\frac{1}{2}\Big[(l_{134} + 2p_{12})V_0^E(p_2, P, \{m\}_{1234}) - S_0^A(p_2, \{m\}_{123}) + S_0^A(P, \{m\}_{124})\Big]. \tag{133}$$

Thus we can write

$$V^E(0\,|\,p_1\,;\,p_2, P, \{m\}_{1234}) = p_1^2\, I_{zy;E} + p_1\cdot P\,\Big[I_{y;E} - V_0^E(p_2, P, \{m\}_{1234})\Big], \tag{134}$$

where two new quantities were introduced,

$$I_{zy;E} = \Gamma(\epsilon)\int \mathcal{D}V_E\,(z-y)\,\chi_E^{-\epsilon}, \qquad I_{y;E} = -\,\Gamma(\epsilon)\int \mathcal{D}V_E\,(1-y)\,\chi_E^{-\epsilon}. \tag{135}$$

Similarly, we derive

$$V^E(0\,|\,p_2\,;\,p_2, P, \{m\}_{1234}) = p_1\cdot p_2\, I_{zy;E} + p_2\cdot P\,\Big[I_{y;E} - V_0^E(p_2, P, \{m\}_{1234})\Big]. \tag{136}$$

Assuming $p_1^2 \neq 0$, we can eliminate one of the two unknowns from Eq.(134) obtaining

$$p_1^2\, I_{zy;E} = -\,p_1\cdot P\,\Big[I_{y;E} - V_0^E(p_2, P, \{m\}_{1234})\Big] + V^E(0\,|\,p_1\,;\,p_2, P, \{m\}_{1234}), \tag{137}$$

and express $V^E(0\,|\,p_2)$ in terms of standard functions and $I_{y;E}$, which is the integral of Eq.(126) with $\alpha = \beta = 2$ and $\gamma = \delta = 1$, corresponding to $n = 6-\epsilon$. Hence, one generalized scalar function in shifted space-time dimension suffices in this class, although we certainly prefer to use $V^E(0|p_2)$ in $n = 4-\epsilon$ dimensions, for which we can derive a smooth integral representation.

9.1.2 Rank two tensor integrals in the V^E family

Tensor integrals with two powers of momenta in the numerator can be treated in a similar way. New ultraviolet divergences arise; for instance with a $q_{1\mu}\, q_{2\nu}$ numerator also the (α, β) sub-diagram is divergent. We define a decomposition according to

$$V^E(\mu, \nu\,|\,0\,;\,\cdots) = V_{111}^E\, p_{1\mu}\, p_{1\nu} + V_{112}^E\, p_{2\mu}\, p_{2\nu} + V_{113}^E\, \{p_1\, p_2\}_{\mu\nu} + V_{114}^E\, \delta_{\mu\nu}, \tag{138}$$

where the list of arguments has been suppressed and where $\{p\,k\}_{\mu\nu}$ is defined in Eq.(13). Strictly analogous definitions hold for the $q_{1\mu}q_{2\nu}$ tensor integrals (V_{12i}^E form factors) and for the $q_2^\mu q_2^\nu$ ones (V_{22i}^E form factors). Consider first the form factors in the $22i$ series; taking the trace in Eq.(138) gives

$$n\, V_{224}^E + p_1^2\, V_{221}^E + 2\, p_{12} V_{223}^E + p_2^2\, V_{222}^E = -\,(p_2^2 + m_3^2)\, V_0^E - 2\,\Big[p_{12}\, V_{21}^E + p_2^2\, V_{22}^E\Big] + S_0^A(P, \{m\}_{124}). \tag{139}$$

As a second step we multiply Eq.(138) by $p_{1\nu}$ and obtain

$$\begin{aligned} V_{224}^E + p_1^2\, V_{221}^E + p_{12}\, V_{223}^E &= \frac{1}{2}\Big[-(l_{134} + 2\,p_{12})\, V_{21}^E + S_0^A(P, \{m\}_{124}) + S_2^A(P, \{m\}_{124})\Big], \\ p_{12}\, V_{222}^E + p_1^2\, V_{223}^E &= \frac{1}{2}\Big[-(l_{134} + 2\,p_{12})\, V_{22}^E \\ &\quad + S_0^A(P, \{m\}_{124}) - S_0^A(p_2, \{m\}_{123}) + S_2^A(P, \{m\}_{124}) - S_2^A(p_2, \{m\}_{123})\Big]. \end{aligned} \tag{140}$$

Eqs.(138)–(140) give a system of three equations for four unknowns; for one of the form factors we can write a combination of two generalized scalar functions, e.g.

$$V^{E}_{224} = \frac{1}{2}\,\omega^2\,\Big[V_{E}^{2|1,1|1} + V_{E}^{1|1,1|2}\Big]\Big|_{n=6-\epsilon}, \tag{141}$$

where ω is defined in Eq.(15), and solve Eqs.(138)–(140) in terms of the generalized scalar functions. An alternative procedure is based on the following integral representations:

$$V^{E}_{22i} = -\Gamma(\epsilon)\int \mathcal{D}V_{E}\,P_{22i;E}\,\chi_{E}^{-\epsilon}(x,y,z), \quad i\neq 4, \qquad V^{E}_{224} = -\frac{1}{2}\,\Gamma(\epsilon-1)\int \mathcal{D}V_{E}\,\chi_{E}^{1-\epsilon}(x,y,z)\,,$$

$$P_{221;E} = z^2, \qquad P_{222;E} = y^2, \qquad P_{223;E} = y\,z. \tag{142}$$

The three integrals can be computed via their representation, following the general strategy already adopted for scalar integrals. Similar representations hold for the remaining tensor integrals, for instance we obtain

$$V^{E}_{124} = -\frac{1}{2}\,\Gamma(\epsilon-1)\int \mathcal{D}V_{E}\,x\,\chi_{E}^{1-\epsilon}(x,y,z),$$

$$V^{E}_{114} = -\,\Gamma(\epsilon)\int \mathcal{D}V_{E}\Big\{-\frac{x(1-x)}{2-\epsilon}\Big[\frac{1}{2}\,\frac{4-\epsilon}{\epsilon-1}\,\chi_{E} + R_{E}\Big] + \frac{1}{2}\,\frac{x^2}{\epsilon-1}\,\chi_{E}\Big\}\,\chi_{E}^{-\epsilon},$$

$$P_{12i;E} = x\,P_{22i;E}, \qquad P_{11i;E} = x^2\,P_{22i;E} \qquad \text{for} \quad i\neq 4, \qquad R_{E} = F(z,y) + m_x^2, \tag{143}$$

with F defined in Eq.(12). Note the singularity hidden in the x-integration in the formulae above. The $q_{1\mu}q_{2\nu}$ tensor integrals are easily reduced as the following relation holds:

$$V^{E}(\mu\,|\,\nu\,;\,p_2,P,\{m\}_{1234}) = \frac{1}{2}\,\Big[m^2_{21}\,V^{I}(0\,|\,\mu\,,\nu\,;\,p_2,P,\{m\}_{12},0,\{m\}_{34}) + V^{E}(0\,|\,\mu\,,\nu\,;\,p_2,P,\{m\}_{1234})$$
$$+\; C_{\mu\nu}(p_2,p_1,0,\{m\}_{34})\,A_0([m_2,m_1])\Big], \tag{144}$$

while those with $q_{1\mu}q_{1\nu}$ require some additional work. To derive the corresponding result we start with

$$\int d^nq_1\,\frac{q_{1\mu}\,q_{1\nu}}{[1][2]_{E}} = B_{22;E}\,\delta_{\mu\nu} + B_{21;E}\,q_{2\mu}\,q_{2\nu}, \tag{145}$$

where the sub-loop form factors are

$$B_{22;E} = \frac{1}{n-1}\,(X_{1;E} - X_{2;E}), \qquad B_{21;E} = \frac{1}{q_2^2}\,(X_{2;E} - B_{22;E}), \tag{146}$$

and also

$$X_{1;E} = \int d^nq_1\,\Big[\frac{1}{[2]_{E}} - \frac{m_1^2}{[1][2]_{E}}\Big], \tag{147}$$

$$X_{2;E} = \frac{1}{4}\int d^nq_1\,\Big[\frac{q_2^2 + 2\,m^2_{21}}{[1][2]_{E}} + \frac{m^4_{12}}{[0]_{E}[1][2]_{E}} + \frac{3}{q_1^2+m_2^2} - \frac{1}{[1]} + m^2_{12}\Big(\frac{1}{[0]_{E}[1]} - \frac{1}{[0]_{E}[2]_{E}}\Big)\Big], \tag{148}$$

with $[0]_{E} = q_2^2$. The complete result reads as follows:

$$V^{E}(\mu,\nu\,|\,0\,;\,\cdots) = \frac{1}{4\,(n-1)}\,\Big[V^{E}_{A}\,\delta_{\mu\nu} + V^{E}_{B,\mu\nu}\Big], \tag{149}$$

$$V^{E}_{A} = -\,m^4_{12}\,V^{I}_{0}(p_2,P,\{m\}_{12},0,\{m\}_{34}) - 2\,(m_1^2+m_2^2)\,V^{E}_{0}(p_2,P,\{m\}_{1234}) - V^{E}(0|\mu,\mu;p_2,P,\{m\}_{1234})$$
$$-\;A_0(m_1)\,\Big[m^2_{21}\,C_0(p_2,p_1,0,\{m\}_{34}) + B_0(\,p_1,\{m\}_{34})\Big]$$
$$-\;A_0(m_2)\,\Big[m^2_{12}\,C_0(p_2,p_1,0,\{m\}_{34}) + B_0(\,p_1,\{m\}_{34})\Big],$$

$$
\begin{aligned}
V^{E}_{B,\mu\nu} &= n\, m^4_{12}\, V^{M}(0\,|\,\mu,\nu\,;\, p_1, P, \{m\}_{12}, 0, \{m\}_{34}, 0) \\
&+ 2\,(n\, m^2_{21} + 2\, m^2_1)\, V^{I}(0\,|\,\mu,\nu\,;\, p_2, P, \{m\}_{12}, 0, \{m\}_{34}) + n\, V^{E}(0\,|\,\mu,\nu\,;\, p_2, P, \{m\}_{1234}) \\
&- n\, A_0(m_1) \Big[m^2_{12}\, C_{\mu\nu}(2,1,1\,;\, p_2, p_1, 0, \{m\}_{34}) - C_{\mu\nu}(p_2, p_1, 0, \{m\}_{34}) \Big] \\
&- A_0(m_2) \Big[(3\,n - 4)\, C_{\mu\nu}(p_2, p_1, 0, \{m\}_{34}) + n\, m^2_{21}\, C_{\mu\nu}(2,1,1\,;\, p_2, p_1, 0, \{m\}_{34}) \Big],
\end{aligned}
\tag{150}
$$

which concludes our analysis of the V^E-family; note that $V^{E}_{B,\mu\nu}$ can be further decomposed following the standard procedure and also contributes to the $\delta_{\mu\nu}$ part of $V^{E}(\mu,\nu\,|\,0\,;\cdots)$. Also for the $q_{2\mu}q_{2\nu}$ tensor integrals we can write down a system of equations and solve it, or we can use their explicit representations. In Eq.(150) we used generalized C-functions; since these functions refer to one-loop diagrams we have full reducibility of tensors while the scalars can be expressed in terms of standard $C_0(1,1,1)$ and $B_0(1,1)$ functions by repeated applications of IBP identities; one should only be aware of the appearance of denominators vanishing at the anomalous threshold. Once again, we could use their explicit parametric representations treated with the BT-algorithm.

Results for this family are summarized in Appendix B.1. $V^{E}_0 \equiv V^{121}_0$ is discussed in Sect. 5.1 of III, the evaluation of the corresponding form factors is addressed in Section 11.1. Note that χ_E (Eq.(125)) is defined in Eq. (62) of III by rescaling by $1/|P^2|$, a normalization which is better suited for numerical integration and that we have used for all the χ functions of III. The same comment (rescaling χ by $1/|P^2|$ in III) applies to all families of diagrams. The ν_i of Eqs. (62) – (63) of III, defined in Eq. (7) of the same paper, coincide with the ν_i quantities defined in Eq.(297) of the present paper.

9.2 The V^I-family ($\alpha = 1, \beta = 3, \gamma = 1$)

We continue our derivation considering the scalar function in the V^I-family of Fig. 7, where only the (α,γ) sub-diagram is ultraviolet divergent. This function is representable as

$$
\pi^4\, V^{I}_0(p_1, P, \{m\}_{12345}) = \mu^{2\epsilon} \int d^n q_1 \int d^n q_2\, \frac{1}{[1][2]_I[3]_I[4]_I[5]_I}, \tag{151}
$$

with propagators

$$
[1] \equiv q^2_1 + m^2_1, \qquad [2]_I \equiv (q_1 - q_2)^2 + m^2_2, \qquad [3]_I \equiv q^2_2 + m^2_3,
$$

$$
[4]_I \equiv (q_2 + p_1)^2 + m^2_4, \quad [5]_I \equiv (q_2 + P)^2 + m^2_5. \tag{152}
$$

Note the symmetry property $V^{I}_0(p_1, P, \{m\}_{12345}) = V^{I}_0(P, p_1, \{m\}_{12354})$, besides the one of Eq.(438).

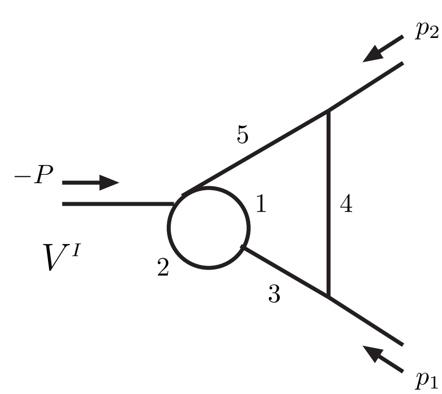


Figure 7: The irreducible two-loop vertex diagrams V^I. External momenta flow inwards. Internal masses are enumerated according to Eq.(152).

9.2.1 Vector integrals in the V^I family

By standard methods we write a decomposition of the vector integrals into form factors

$$
\begin{aligned}
V^I(\mu\,|\,0\,;\,p_1,P,\{m\}_{12345}) &= \sum_{i=1,2} V^I_{1i}(p_1,P,\{m\}_{12345})\,p_{i\mu},\\
V^I(0\,|\,\mu\,;\,p_1,P,\{m\}_{12345}) &= \sum_{i=1,2} V^I_{2i}(p_1,P,\{m\}_{12345})\,p_{i\mu}.
\end{aligned}
\tag{153}
$$

As we mentioned earlier, special care must be used when V^I_{ij} appears in the reduction of other form factors and one has to bring the integrand in a form adhering to Eq.(151); this can be done using the definition,

$$
\frac{\mu^{2\epsilon}}{\pi^4}\int d^n r_1\, d^n r_2\, r_{i\mu}\prod_{l=a}^{e} D_l^{-1} \equiv V^I_{i1}(k_d,k_e,\{m\}_{abcde})\,k_{d\mu} + V^I_{i2}(k_d,k_e,\{m\}_{abcde})\,(k_e-k_d)_\mu\,,
\tag{154}
$$

where, with an obvious notation

$$
\begin{aligned}
D_a &= r_1^2+m_a^2\,, & D_b &= (r_1-r_2)^2+m_b^2\,, & D_c &= r_2^2+m_c^2\,,\\
D_d &= (r_2+k_d)^2+m_d^2\,, & D_e &= (r_2+k_e)^2+m_e^2.
\end{aligned}
$$

Note that the following identities hold:

$$
V^I_{i1}(d,e) = V^I_{i1}(e,d), \qquad V^I_{i2}(d,e) = V^I_{i1}(e,d) - V^I_{i2}(e,d),
\tag{155}
$$

where $(d,e)=(k_d,k_e,\{m\}_{abcde})$ etc. The explicit expression for the form factors of Eq.(153) is

$$
\begin{aligned}
V^I_{ij} &= -\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_I\, P_{ij;I}\,\chi_I^{-1-\epsilon},\\
P_{00;I} &= 1, \quad P_{1i;I} = x\,P_{2i;I}, \quad P_{21;I} = -z_1, \quad P_{22;I} = -z_2,
\end{aligned}
\tag{156}
$$

where P_{00} is the factor corresponding to the scalar integral and, with our choice for the Feynman parameters, the polynomial χ_I is

$$
\chi_I(x,y,z_1,z_2) = -\,F(z_1,z_2) \,+\, l_{134}\,z_1 \,+\, (l_{245}\,+\,2\,p_{12})\,z_2 \,+\, (m_3^2\,-\,m_x^2)\,y \,+\, m_x^2,
\tag{157}
$$

where F and m_x^2 are defined in Eq.(12). Finally, the integration measure is

$$
\int \mathcal{D}V_I = \omega^\epsilon \int dCS\,(x\,;\,y,z_1,z_2)\,\Big[x\,(1-x)\Big]^{-\epsilon/2}(1-y)^{\epsilon/2-1},
\tag{158}
$$

with ω defined in Eq.(15). All these functions can be manipulated according to the procedure introduced in III with correspondingly smooth integral representations.

This family is the first example of a vertex with full q_2 reducibility. Consider the q_1 vector integral: for the case $m_3\neq 0$ and by methods similar to the ones used in Section 9.1 for V^E we obtain

$$
\begin{aligned}
V^I(\mu\,|\,0\,;\,p_1,P,\{m\}_{12345}) &= \frac{1}{2}\frac{m^2_{123}}{m_3^2}\,V^I(0\,|\,\mu\,;\,p_1,P,\{m\}_{12345}) - \frac{1}{2}\frac{m^2_{12}}{m_3^2}\,V^I(0\,|\,\mu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45})\\
&\quad - \frac{1}{2m_3^2}\,A_0([m_1,m_2])\Big[C_\mu(p_1,p_2,\{m\}_{345}) - C_\mu(p_1,p_2,0,\{m\}_{45})\Big].
\end{aligned}
\tag{159}
$$

Furthermore, the q_2 vector integral can be reduced according to the following relation:

$$
\begin{aligned}
V^I(0|p_1;p_1,P,\{m\}_{12345}) &= \frac{1}{2}\Big[-l_{134}V^I_0(p_1,P,\{m\}_{12345}) - V^E_0(p_1,P,\{m\}_{1245}) + V^E_0(0,P,\{m\}_{1235})\Big],\\
V^I(0|p_2;p_1,P,\{m\}_{12345}) &= \frac{1}{2}\Big[(l_{154}-P^2)V^I_0(p_1,P,\{m\}_{12345}) + V^E_0(0,p_1,\{m\}_{1234}) - V^E_0(0,P,\{m\}_{1235})\Big].
\end{aligned}
\tag{160}
$$

9.2.2 Rank two tensor integrals in the V^I family

All tensor integrals in this class are overall ultraviolet divergent, with a divergent (α, γ) sub-diagram. Adopting the same notation employed in the analysis of the V^E functions, we introduce the form factors V^{I}_{ijm}, $(m = 1, \dots, 4)$:

$$V^I(0|\mu,\nu\,;\,\cdots) = V^{I}_{221}\, p_{1\mu}\, p_{1\nu} + V^{I}_{222}\, p_{2\mu}\, p_{2\nu} + V^{I}_{223}\, \{p_1 p_2\}_{\mu\nu} + V^{I}_{224}\, \delta_{\mu\nu}, \tag{161}$$

where the symmetrized product is given by Eq.(13). Taking the trace of both sides in Eq.(161) gives

$$p_1^2\, V^{I}_{221} + 2\, p_{12}\, V^{I}_{223} + p_2^2\, V^{I}_{222} + \, n\, V^{I}_{224} = V^{E}_{0}(p_1, P, \{m\}_{1245}) - m_3^2\, V^{I}_{0}(p_1, P, \{m\}_{12345}), \tag{162}$$

while, multiplying both sides of Eq.(161) by $p_{1\nu}$, we have the relations

$$\begin{aligned} p_1^2 V^{I}_{221} + p_{12} V^{I}_{223} + V^{I}_{224} &= \frac{1}{2}\Big[-V^{E}_{21}(p_1, P, \{m\}_{1245}) + V^{E}_{22}(0, P, \{m\}_{1235}) - l_{134} V^{I}_{22}(p_1, P, \{m\}_{12345})\Big], \\ p_{12} V^{I}_{222} + p_1^2 V^{I}_{223} &= \frac{1}{2}\Big[-V^{E}_{21}(p_1, P, \{m\}_{1245}) + V^{E}_{21}(0, P, \{m\}_{1235}) - l_{134} V^{I}_{21}(p_1, P, \{m\}_{12345})\Big]. \end{aligned} \tag{163}$$

Similarly, contracting Eq.(161) with $p_{2\nu}$, it is possible to write additional identities:

$$\begin{aligned} p_{12}\, V^{I}_{221} + p_2^2\, V^{I}_{223} &= -\,\frac{1}{2}\,\Big[V^{E}_{21}(0, P, \{m\}_{1235}) - \, V^{E}_{21}(0, p_1, \{m\}_{1234}) + \, (l_{P45} - p_1^2)\, V^{I}_{21}(p_1, P, \{m\}_{12345})\Big], \\ p_2^2\, V^{I}_{222} + p_{12}\, V^{I}_{223} + V^{I}_{224} &= \frac{1}{2}\,\Big[-\, V^{E}_{21}(0, P, \{m\}_{1235}) - \, (l_{P45} - p_1^2)\, V^{I}_{22}(p_1, P, \{m\}_{12345})\Big]. \end{aligned} \tag{164}$$

Solving the system formed by Eqs.(162)–(164) it is then possible to express the V^{I}_{22i} form factor in terms of functions V^{I}_{2i} and form factors belonging to the V^E family.

The integral representation for the V^{I}_{22i} functions is the following:

$$\begin{aligned} V^{I}_{22i} &= -\,\Gamma\,(1+\epsilon) \int \mathcal{D}V_I\, R_{22i;I}\, \chi_I^{-1-\epsilon}, \qquad V^{I}_{224} = -\frac{1}{2}\,\Gamma\,(\epsilon) \int \mathcal{D}V_I\, \chi_I^{-\epsilon}, \\ R_{221;I} &= z_1^2, \quad R_{222;I} = z_2^2, \quad R_{223;I} = z_1\, z_2, \end{aligned} \tag{165}$$

showing for instance a double ultraviolet pole for V^{I}_{224}. Similar integral representations can be found for the form factors V^{I}_{12i}:

$$V^{I}_{12i} = -\,\Gamma\,(1+\epsilon) \int \mathcal{D}V_I\, R_{12i;I}\, \chi_I^{-1-\epsilon}, \quad V^{I}_{124} = -\frac{1}{2}\,\Gamma\,(\epsilon) \int \mathcal{D}V_I\, x\, \chi_I^{-\epsilon}, \quad R_{12i;I} = x\, R_{22i;I}. \tag{166}$$

The $q_{1\mu}\, q_{2\nu}$ or $q_{1\nu}\, q_{2\mu}$ tensor integrals can be written in terms of form factors V^{I}_{22i} employing the following relation, valid for $m_3 \neq 0$:

$$\frac{1}{[0]_I [3]_I} = \frac{1}{m_3^2}\,\Big(\frac{1}{[0]_I} - \frac{1}{[3]_I}\Big), \tag{167}$$

where $[0]_I = q_2^2$; in this way one obtains

$$\begin{aligned} V^I(\mu|\nu\,;\, p_1, P, \{m\}_{12345}) &= \frac{m_{123}^2}{2m_3^2}\, V^I(0|\mu,\nu\,;\, p_1, P, \{m\}_{12345}) + \frac{m_{21}^2}{2m_3^2}\, V^I(0|\mu,\nu\,;\, p_1, P, \{m\}_{12}, 0, \{m\}_{45}) \\ &- \, \frac{1}{2m_3^2}\, A_0([m_1, m_2])\Big[\, C_{\mu\nu}(p_1, p_2, \{m\}_{345}) - C_{\mu\nu}(p_1, p_2, 0, \{m\}_{45})\Big]. \end{aligned} \tag{168}$$

The integral representation of the V^{I}_{11i} form factors is

$$\begin{aligned} V^{I}_{11i} &= -\,\Gamma\,(1+\epsilon) \int \mathcal{D}V_I\, R_{11i;I}\, \chi_I^{-1-\epsilon}, \\ V^{I}_{114} &= -\,\Gamma\,(\epsilon) \int \mathcal{D}V_I\, \chi_I^{-\epsilon}\, \Big\{-\frac{x(1-x)}{2-\epsilon}\,\Big[\frac{4-\epsilon}{2} + \epsilon\, \chi_I^{-1}\, R_I\Big] + \, \frac{1}{2}\, x^2\,\Big\}, \\ R_{11i;I} &= x^2\, R_{22i;I}, \qquad R_I = F(z_1, z_2) + m_x^2, \end{aligned} \tag{169}$$

with F defined in Eq.(12), The form factors V^{I}_{11i} can be reduced using q_1 sub-loop techniques, similarly to what we did for the V^{E}_{11i} functions, and employing Eq.(167). One obtains

$$V^{I}(\mu,\nu\,|\,0\,;\,\cdots) = \frac{1}{4\,(n-1)}\,\Big[V^{I}_{A}\,\delta_{\mu\nu} + V^{I}_{B,\mu\nu}\Big], \tag{170}$$

$$\begin{aligned}
V^{I}_{A} = {}& \frac{1}{m_3^2}\Big\{-A_0(m_1)\Big[m^2_{123}\,C_0(p_1,p_2,\{m\}_{345}) + m^2_{21}\,C_0(p_1,p_2,0,\{m\}_{45})\Big] \\
& - A_0(m_2)\Big[m^2_{213}\,C_0(p_1,p_2,\{m\}_{345}) + m^2_{12}\,C_0(p_1,p_2,0,\{m\}_{45})\Big] \\
& - m_3^2\,V^{I}(0|\mu,\mu\,;\,p_1,P,\{m\}_{12345}) + \Big[m^4_{12} - 2\,m_3^2\,(m_1^2+m_2^2)\Big]\,V^{I}_{0}(p_1,P,\{m\}_{12345}) \\
& - m^4_{12}\,V^{I}_{0}(p_1,P,\{m\}_{12},0,\{m\}_{45})\Big\}, \\
V^{I}_{B,\mu\nu} = {}& \frac{1}{m_3^2}\Big\{n\,m_3^2\,m^4_{12}\Big[V^{M}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45},0) - V^{M}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345},0)\Big] \\
& - n\,m_3^2\,A_0(m_1)\Big[C_{\mu\nu}\,(p_1,p_2,\{m\}_{345}) - C_{\mu\nu}\,(p_1,p_2,0,\{m\}_{45})\Big] \\
& - n\,m^2_{12}\,A_0([m_1,m_2])\Big[m_3^2\,C_{\mu\nu}(2,1,1\,;\,p_1,p_2,0,\{m\}_{45}) - C_{\mu\nu}\,(p_1,p_2,0,\{m\}_{45}) \\
& + C_{\mu\nu}\,(p_1,p_2,\{m\}_{345})\Big] + (3\,n-4)\,A_0(m_2)\Big[C_{\mu\nu}\,(p_1,p_2,\{m\}_{345}) - C_{\mu\nu}\,(p_1,p_2,0,\{m\}_{45})\Big] \\
& + m_3^2\,(2\,n\,m^2_{12} + n\,m_3^2 - 4\,m_1^2)\,V^{I}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345}) \\
& - m_3^2\Big[(n-4)\,m_1^2 - 2\,n\,m_2^2\Big]\,V^{I}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45})\Big\}\,.
\end{aligned} \tag{171}$$

Note that $V^{I}_{B,\mu\nu}$ will be further decomposed into $\delta_{\mu\nu}$ and $p_{i\mu}p_{j\nu}$ terms. Results for this family are summarized in Appendix B.2. $V^{I}_{0} \equiv V^{131}_{0}$ is discussed in Sect. 6.1 of III (see comment at the end of Section 9.1.2), evaluation of form factors in Section 11.2. Note that $\chi_{I}(x,1,y,z)$ does not depend on x and, in III, we used $\chi_{I}(y,z) \equiv \chi_{I}(x,1,y,z)$. In the following Section we move to the discussion of the V^{M} class of diagrams.

9.3 The V^{M}-family ($\alpha = 1, \beta = 4, \gamma = 1$)

The scalar V^{M} function of Fig. 8 is overall ultraviolet convergent with the (α,γ) sub-diagram divergent and is representable as follows:

$$\pi^4\,V^{M}_{0}(p_1,P,\{m\}_{123456}) = \mu^{2\epsilon}\int d^nq_1\int d^nq_2\,\frac{1}{[1][2]_{M}[3]_{M}[4]_{M}[5]_{M}[6]_{M}}, \tag{172}$$

with propagators

$$[1] \equiv q_1^2 + m_1^2, \qquad [2]_{M} \equiv (q_1-q_2)^2 + m_2^2, \qquad [3]_{M} \equiv q_2^2 + m_3^2,$$

$$[4]_{M} \equiv (q_2+p_1)^2 + m_4^2, \qquad [5]_{M} \equiv (q_2+P)^2 + m_5^2, \qquad [6]_{M} \equiv q_2^2 + m_3^2\,. \tag{173}$$

Note the symmetry property $V^{M}_{0}(p_1,P,\{m\}_{123456}) = V^{M}_{0}(P,p_1,\{m\}_{123546})$, as shown in Eq.(438) of Appendix C. Scalar, vector and rank two tensor integrals have an (α,γ) sub-diagram which is ultraviolet divergent. As it was pointed out in III, we need to consider just the case $m_3 = m_6$ and, as a consequence, m_6 drops from the list of arguments; in fact, when these two masses are different it is possible to rewrite the integral as a difference of V^{I}-functions.

9.3.1 Vector integrals in the V^{M} family

As usual, we introduce form factors for the vector integrals according to the equations

$$V^{M}(\mu\,|\,0\,;\,p_1,P,\{m\}_{12345}) = \sum_{i=1,2} V^{M}_{1i}(p_1,P,\{m\}_{12345})\,p_{i\mu},$$

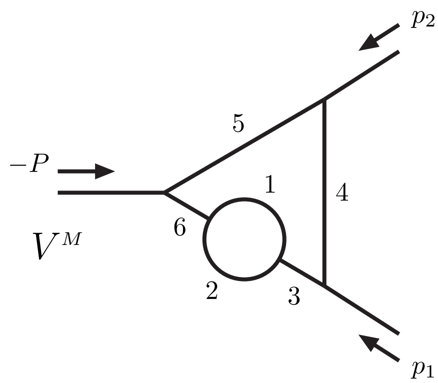


Figure 8: The irreducible two-loop vertex diagrams V^M. External momenta flow inwards. Internal masses are enumerated according to Eq.(173).

$$V^M(0\,|\,\mu\,;\,p_1,P,\{m\}_{12345}) = \sum_{i=1,2} V^M_{2i}(p_1,P,\{m\}_{12345})\,p_{i\mu}, \tag{174}$$

where the form factors $V^M_{ij}(p,k,\{m\}_{a\cdots e})$ refer to the basis p_μ and $(k-p)_\mu$.

The integral representation of the form factors introduced in Eq.(174) is obtained employing standard methods:

$$V^M_{ij} = -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_M\,P_{ij;M}\,\chi_M^{-2-\epsilon}, \tag{175}$$

$$P_{00;M} = 1, \quad P_{1i;M} = x\,P_{2i;M}, \quad P_{21;M} = -z_1, \quad P_{22;M} = -z_2, \tag{176}$$

where the integration measure is

$$\int \mathcal{D}V_M = \omega^\epsilon \int dCS\,(x\,;\,y,z_1,z_2)\,(y-z_1)\Big[x\,(1-x)\Big]^{-\epsilon/2}(1-y)^{\epsilon/2-1}, \tag{177}$$

and ω is defined in Eq.(15); P_{00} is the factor corresponding to the integral representation of the scalar function. The polynomial χ_M is equal to χ_I.

It is possible to rewrite the q_1 vector integral in terms of the q_2 vector integral; when $m_3 \neq 0$ one finds

$$\begin{aligned} V^M(\mu|0\,;\,p_1,P,\{m\}_{12345}) &= \frac{m^2_{123}}{2m_3^2}\,V^M(0\,|\,\mu\,;\,p_1,P,\{m\}_{12345}) + \frac{m^2_{12}}{2m_3^4}\,[V^I(0\,|\,\mu\,;\,p_1,P,\{m\}_{12345}) \\ &- V^I(0\,|\,\mu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45})] - \frac{A_0([m_1,m_2])}{2m_3^4}\Big[C_\mu(2,1,1\,;\,p_1,p_2,\{m\}_{345})\,m_3^2 \\ &+ C_\mu(p_1,p_2,\{m\}_{345}) - C_\mu(p_1,p_2,0,\{m\}_{45})\Big]. \end{aligned} \tag{178}$$

The q_2 vector integrals can be reduced to a linear combination of scalar factors as follows:

$$\begin{aligned} V^M(0|p_1\,;\,p_1,P,\{m\}_{12345}) &= \frac{1}{2}\Big[-V^M_0\,(p_1,P,\{m\}_{12345})\,l_{134} \\ &\quad - V^I_0\,(p_1,P,\{m\}_{12345}) + V^I_0\,(0,P,\{m\}_{12335})\Big], \\ V^M(0|p_2\,;\,p_1,P,\{m\}_{12345}) &= \frac{1}{2}\Big[V^M_0\,(p_1,P,\{m\}_{12345})\,(l_{154}-P^2) \\ &\quad + V^I_0\,(0,p_1,\{m\}_{12334}) - V^I_0\,(0,P,\{m\}_{12335})\Big]. \end{aligned} \tag{179}$$

9.3.2 Rank two tensor integrals in the V^M family

The tensor integrals with two powers of the integration momenta in the numerator can be treated analogously to the case of V^I. Only the (α, γ) sub-diagram is ultraviolet divergent. Using Eq.(13) we define the corresponding decomposition as

$$V^M(0|\mu,\nu\,;\,\cdots) = V^M_{221}\,p_{1\mu}\,p_{1\nu} + V^M_{222}\,p_{2\mu}\,p_{2\nu} + V^M_{223}\,\{p_1p_2\}_{\mu\nu} + V^M_{224}\,\delta_{\mu\nu}, \tag{180}$$

and the corresponding form factors V^M_{11i} and V^M_{12i}. The symmetrized product is given by Eq.(13). Consider the form factors of the V_{22i} family: taking the trace in both sides of Eq.(180) one obtains the relation

$$p_1^2\,V^M_{221} + 2\,p_{12}\,V^M_{223} + p_2^2\,V^M_{222} + n\,V^M_{224} = V^I_0(p_1,P,\{m\}_{12345}) - m_3^2\,V^M_0(p_1,P,\{m\}_{12345})\,. \tag{181}$$

Similarly, contracting Eq.(180) with $p_{1\nu}$ we have

$$p_1^2V^M_{221} + p_{12}V^M_{223} + V^M_{224} = \frac{1}{2}\Big[-V^I_{21}(p_1,P,\{m\}_{12345}) + V^I_{22}(0,P,\{m\}_{12335}) - l_{134}V^M_{21}(p_1,P,\{m\}_{12345})\Big], \tag{182}$$

$$p_{12}V^M_{222} + p_1^2V^M_{223} = \frac{1}{2}\Big[-V^I_{22}(p_1,P,\{m\}_{12345}) + V^I_{22}(0,P,\{m\}_{12335}) - l_{134}V^M_{22}(p_1,P,\{m\}_{12345})\Big]. \tag{183}$$

Contracting Eq.(180) with $p_{2\nu}$ we get instead

$$p_{12}V^M_{221} + p_2^2V^M_{223} = \frac{1}{2}\Big[V^I_{22}(0,p_1,\{m\}_{12334}) - V^I_{22}(0,P,\{m\}_{12335}) + (p_1^2 - l_{P45})V^M_{21}(p_1,P,\{m\}_{12345})\Big],$$

$$p_2^2V^M_{222} + p_{12}V^M_{223} + V^M_{224} = \frac{1}{2}\Big[-V^I_{22}(0,P,\{m\}_{12335}) + (p_1^2 - l_{P45})V^M_{22}(p_1,P,\{m\}_{12345})\Big]. \tag{184}$$

Solving the system given by Eqs.(181)–(184), we can reduce the V^M_{22i} form factors to linear combinations of vector and scalar integrals. The integral representation of these form factors is the following:

$$V^M_{22i} = -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_M\,R_{22i;M}\,\chi_I^{-2-\epsilon}, \quad i \neq 4\,, \qquad V^M_{224} = -\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_M\,\chi_M^{-1-\epsilon},$$

$$R_{221;M} = z_1^2, \quad R_{222;M} = z_2^2, \quad R_{223;M} = z_1\,z_2. \tag{185}$$

The form factors of the V^M_{12i} family can be written in terms of those of the V^M_{22i} family: in fact we have that

$$\begin{aligned}V^M(\mu|\nu\,;\,p_1,P,\{m\}_{12345}) = {} & V^M(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345})\,\frac{m^2_{312}}{2m_3^2} + \frac{m^2_{12}}{2\,m_3^4}\Big[V^I(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345}) \\ & - V^I(0|\mu,\nu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45})\Big] \;-\; \frac{A_0([m_1,m_2])}{2m_3^4}\Big[C_{\mu\nu}(p_1,p_2,\{m\}_{345}) \\ & - C_{\mu\nu}(p_1,p_2,0,\{m\}_{45}) + m_3^2\,C_{\mu\nu}(2,1,1\,;\,p_1,p_2,\{m\}_{345})\Big].\end{aligned} \tag{186}$$

The integral representation of the same form factors is the following:

$$V^M_{12i} = -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_M\,R_{12i;M}\,\chi_M^{-2-\epsilon}, \quad i \neq 4, \quad V^M_{124} = -\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_M\,x\,\chi_M^{-1-\epsilon}, \tag{187}$$

with $R_{12i;M} = x\,R_{22i;M}$. Similarly to the case of the V^E and V^I families, the reduction of $q_{1\mu}q_{1\nu}$ tensor integrals leads to expressions which are more involved. Introducing the definitions

$$V^M(\mu,\nu\,|\,0\,;\,\cdots) = \frac{1}{4\,(n-1)}\Big[V^M_A\,\delta_{\mu\nu} + V^M_{B,\,\mu\nu}\Big], \tag{188}$$

and employing standard techniques one finds

$$
\begin{aligned}
V^{M}_{A} &= \frac{1}{m_3^4}\Bigl\{V^{M}_{0}(p_1,P,\{m\}_{12345})\,m_3^2\Bigl[m_{12}^4 - 2\,(m_1^2+m_2^2)\,m_3^2\Bigr] - m_3^4\,V^{M}(0|\mu,\mu\,;\,p_1,P,\{m\}_{12345})\\
&\quad + m_{12}^4\Bigl[V^{I}_{0}(p_1,P,\{m\}_{12345}) - V^{I}_{0}(p_1,P,\{m\}_{12},0,\{m\}_{45})\Bigr]\\
&\quad - m_{12}^2\,A_0([m_1,m_2])\Bigl[C_0(p_1,p_2,\{m\}_{345}) - C_0(p_1,p_2,0,\{m\}_{45})\Bigr]\\
&\quad - m_3^2\,C_0(2,1,1\,;\,p_1,p_2,\{m\}_{345})\Bigl[m_{123}^2\,A_0(m_1) + m_{213}^2\,A_0(m_2)\Bigr]\Bigr\},\\
V^{M}_{B,\,\mu\nu} &= \frac{1}{m_3^4}\Bigl\{(n\,m_{123}^4 - 4\,m_1^2\,m_3^2)\,V^{M}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345}) + V^{M}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45})\,n\,m_{12}^4\\
&\quad + \frac{2}{m_3^2}\,(2\,m_1^2\,m_3^2 - n\,m_{12}^2\,m_{123}^2)\Bigl[V^{I}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12},0,\{m\}_{45}) - V^{I}(0|\mu,\nu\,;\,p_1,P,\{m\}_{12345})\Bigr]\\
&\quad + n\Bigl(2\,\frac{m_{123}^2}{m_3^2}-1\Bigr)\,A_0([m_1,m_2])\Bigl[C_{\mu\nu}(p_1,p_2,\{m\}_{345}) - C_{\mu\nu}(p_1,p_2,0,\{m\}_{45})\Bigr]\\
&\quad - n\,A_0([m_1,m_2])\Bigl[m_{123}^2\,C_{\mu\nu}(2,1,1\,;\,p_1,p_2,\{m\}_{345}) + m_{12}^2\,C_{\mu\nu}(2,1,1\,;\,p_1,p_2,0,\{m\}_{45})\Bigr]\\
&\quad + 2\,(n-2)\,A_0(m_2)\Bigl[C_{\mu\nu}(p_1,p_2,\{m\}_{345}) - C_{\mu\nu}(p_1,p_2,0,\{m\}_{45})\\
&\quad + m_3^2\,C_{\mu\nu}(2,1,1\,;\,p_1,p_2,\{m\}_{345})\Bigr\}. \qquad (189)
\end{aligned}
$$

Note that $V^{M}_{B,\mu\nu}$ will be further decomposed into $\delta_{\mu\nu}$ and $p_{i\mu}p_{j\nu}$ terms. The integral representations for the form factor V^{M}_{11i} are the following:

$$
\begin{aligned}
V^{M}_{11i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{M}\,R_{12i;M}\,\chi_{M}^{-2-\epsilon}, \quad i \neq 4\,,\\
V^{M}_{114} &= -\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{M}\Bigl\{-\frac{x(1-x)}{2-\epsilon}\Bigl[\frac{4-\epsilon}{2} + (1+\epsilon)\,\chi_{M}^{-1}\,R_{M}\Bigr] + \frac{1}{2}\,x^2\Bigr\}\,\chi_{M}^{-1-\epsilon},\\
R_{11i;M} &= x^2\,R_{22i;M}, \qquad R_{M} = F(z_1,z_2) + m_x^2, \qquad (190)
\end{aligned}
$$

with F and m_x^2 defined in Eq.(12). Consider now the generalized $V^{\alpha_1|\alpha_2,\alpha_3,\alpha_4|\alpha_5}$ function where the propagator carrying mass m_i is raised to the α_i power:

$$
\begin{aligned}
V^{\alpha_1|\alpha_3,\alpha_4,\alpha_5|\alpha_2}_{M}(n) &= -\frac{\Gamma\,(2+\epsilon)}{\Pi_{i=1}^{5}\,\Gamma\,(\alpha_i)}\,\omega^{6-\sum_{j=1}^{5}\alpha_j+\epsilon}\int dCS\,(x\,;\,y,z_1,z_2)\\
&\quad\times (1-y)^{\rho_1}\,(y-z_1)^{\rho_2}\,(z_1-z_2)^{\rho_3}\,x^{\rho_4}\,(1-x)^{\rho_5}\,z_2^{\rho_6}\,\chi_{M}^{-2-\epsilon}. \qquad (191)
\end{aligned}
$$

The space-time dimension is $n = \sum_{i=1}^{5}\alpha_i - 2 - \epsilon$ and the various powers appearing in Eq.(191) are:

$$
\begin{aligned}
&\rho_1 = \alpha_1 + \alpha_2 - \frac{1}{2}\,(\sum\alpha - \epsilon), \qquad \rho_2 = \alpha_3 - 1, \qquad \rho_3 = \alpha_4 - 1,\\
&\rho_4 = \frac{1}{2}\sum\alpha - \alpha_1 - 2 - \frac{1}{2}\,\epsilon, \qquad \rho_5 = \frac{1}{2}\sum\alpha - \alpha_2 - 2 - \frac{1}{2}\,\epsilon, \qquad \rho_6 = \alpha_5 - 1. \qquad (192)
\end{aligned}
$$

ω is defined in Eq.(15). Results for this family are summarized in Appendix B.3. $V^{M}_{0} \equiv V^{141}_{0}$ is discussed in Sects. 8.1–8.2 of III (see comment at the end of Section 9.1.2), evaluation of form factors in Section 11.2.

9.4 The V^{G}-family ($\alpha = 2, \beta = 2, \gamma = 1$)

We continue our analysis considering the scalar function in the V^{G}-family of Fig. 9 which is ultraviolet convergent with all its sub-diagrams and is representable as

$$
\pi^4\,V^{G}_{0}(p_1,p_1,P,\{m\}_{12345}) = \mu^{2\epsilon}\int d^nq_1\int d^nq_2\,\frac{1}{[1][2]_G[3]_G[4]_G[5]_G}, \qquad (193)
$$

with propagators

$$[1] \equiv q_1^2 + m_1^2, \quad [2]_G \equiv (q_1 + p_1)^2 + m_2^2, \quad [3]_G \equiv (q_1 - q_2)^2 + m_3^2,$$

$$[4]_G \equiv (q_2 + p_1)^2 + m_4^2, \quad [5]_G \equiv (q_2 + P)^2 + m_5^2. \tag{194}$$

This family represents the first case where the scalar configuration is ultraviolet finite while tensor integrals are divergent.

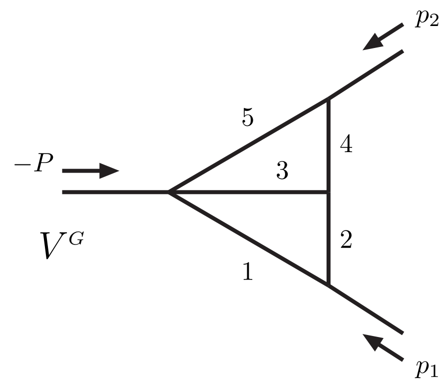


Figure 9: The irreducible two-loop vertex diagrams V^G. External momenta flow inwards. Internal masses are enumerated according to Eq.(194).

9.4.1 *Vector integrals in the V^G family*

Decomposition of vector integrals follows in the usual way:

$$V^G(\mu \,|\, 0\,;\, p_1, p_1, P, \{m\}_{12345}) = \sum_{i=1}^{2} V^G_{1i}(p_1, p_1, P, \{m\}_{12345})\, p_{i\mu},$$

$$V^G(0 \,|\, \mu\,;\, p_1, p_1, P, \{m\}_{12345}) = \sum_{i=1}^{2} V^G_{2i}(p_1, p_1, P, \{m\}_{12345})\, p_{i\mu}, \tag{195}$$

where the form factors $V^G_{ij}(p, p, k, \{m\}_{a\cdots e})$ refer to the basis p_μ and $(k-p)_\mu$. Their explicit expression is

$$\begin{aligned}
V^G_{ij} &= -\,\Gamma\,(1+\epsilon) \int \mathcal{D}V_G\, P_{ij;G}\, \chi_G^{-1-\epsilon}, \\
\int \mathcal{D}V_G &= \omega^\epsilon \int dS_2(\{x\}) \int dS_2(\{y\}) \Big[x_2\,(1-x_2)\Big]^{-1-\epsilon/2} y_2^{\epsilon/2}, \\
P_{00;G} &= 1, \quad P_{11;G} = -1 + x_1 - x_2\,(1-y_2) - x_2\, y_2\, X, \quad P_{12;G} = x_2\,(y_1 - 1), \\
P_{21;G} &= -1 + y_2\,\overline{X}, \quad P_{22;G} = y_1 - 1,
\end{aligned} \tag{196}$$

where ω is defined in Eq.(15) and $X = (1-x_1)/(1-x_2) = 1 - \overline{X}$. The polynomial χ_G is given by

$$\begin{aligned}
\chi_G = [x_2(1-x_2)]^{-1} \Big\{ &-\frac{1}{\overline{x}_2}\, F(\overline{x}\, y_2\,,\, \overline{x}_2\, y_1) + \overline{x}_2\, l_{254}\, y_1 \\
&+ \Big[\overline{x}_2\,(M_x^2 - m_4^2 - p_2^2 + P^2) - \overline{x}_1\,(p_1^2 - p_2^2 + P^2)\Big]\, y_2 + \overline{x}_2\, M_x^2 \Big\},
\end{aligned} \tag{197}$$

with F defined in Eq.(12) and

$$x_2\, \overline{x}_2\, M_x^2 = \overline{x}\, m_1^2 + \overline{x}_1\, m_2^2 + x_2\, m_3^2 + x_1\, \overline{x}_1\, p_1^2, \tag{198}$$

where $\overline{x}_i = 1-x_i$, $\overline{x} = x_1-x_2$. All these functions can be manipulated according to the procedure introduced in III. The generic scalar function in this family is

$$V_{\scriptscriptstyle G}^{\alpha_1,\alpha_2|\alpha_3,\alpha_4|\alpha_5}(n) = -\,\frac{\Gamma\,(1+\epsilon)}{\prod_{i=1}^5 \Gamma\,(\alpha_i)}\,\omega^{5-\sum_{j=1}^5 \alpha_j+\epsilon} \int dS_2(\{x\}) \int dS_2(\{y\})\,\chi_{\scriptscriptstyle G}^{-1-\epsilon}$$
$$\times\,(x_1-x_2)^{\rho_1}\,(1-x_1)^{\rho_2}\,(y_1-y_2)^{\rho_3}\,(1-y_1)^{\rho_4}\,(1-x_2)^{\rho_5}\,y_2^{\rho_6}\,x_2^{\rho_7}, \tag{199}$$

with ω defined in Eq.(15), the dimension $n = \sum_{i=1}^5 \alpha_i - 1 - \epsilon$ and powers

$$\rho_i = \alpha_i - 1, \quad i = 1,\cdots,4, \qquad \rho_5 = \frac{1}{2}\sum\alpha - \alpha_5 - \alpha_1 - \alpha_2 - \frac{1}{2}\,(1+\epsilon),$$
$$\rho_6 = \alpha_5 + \alpha_1 + \alpha_2 - \frac{1}{2}\,(\sum\alpha + 1 - \epsilon), \qquad \rho_7 = \frac{1}{2}\sum\alpha - \alpha_1 - \alpha_2 - \frac{1}{2}\,(3+\epsilon). \tag{200}$$

Also for this case we have partial reducibility,

$$V^{\scriptscriptstyle G}(p_1\,|\,0\,;\,p_1,p_1,P,\{m\}_{12345}) = \frac{1}{2}\Big[-\,l_{112}\,V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(p_1,p_1,P,\{m\}_{12345}) + V_{\scriptscriptstyle 0}^{\scriptscriptstyle E}(p_1,P,\{m\}_{1345})$$
$$-\,V_{\scriptscriptstyle 0}^{\scriptscriptstyle E}(0,p_2,\{m\}_{2345})\Big],$$
$$V^{\scriptscriptstyle G}(0\,|\,p_2\,;\,p_1,p_1,P,\{m\}_{12345}) = \frac{1}{2}\Big[(-l_{245}-2\,p_{12})\,V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(p_1,p_1,P,\{m\}_{12345}) -\,V_{\scriptscriptstyle 0}^{\scriptscriptstyle E}(-p_2,-P,\{m\}_{5321})$$
$$+\,V_{\scriptscriptstyle 0}^{\scriptscriptstyle E}(0,-p_1,\{m\}_{4321})\Big]. \tag{201}$$

Note that $V_{\scriptscriptstyle 0}^{\scriptscriptstyle E}(0,p)$ is equivalent to two-point functions of the $S^{\scriptscriptstyle C}$ family, Eq.(80). The system of equations that we obtain is

$$V^{\scriptscriptstyle G}(p_1\,|\,0) = p_1^2\,V_{\scriptscriptstyle 11}^{\scriptscriptstyle G} + p_{12}\,V_{\scriptscriptstyle 12}^{\scriptscriptstyle G}, \qquad V^{\scriptscriptstyle G}(0\,|\,p_2) = p_{12}\,V_{\scriptscriptstyle 21}^{\scriptscriptstyle G} + p_2^2\,V_{\scriptscriptstyle 22}^{\scriptscriptstyle G}. \tag{202}$$

Assuming that $p_1^2 \neq 0$ we can eliminate the integral with $P_{\scriptscriptstyle 11}$ in Eq.(196) in favor of the integral with $P_{\scriptscriptstyle 12}$ which contains the factor $x_2(y_1-1)$ and obtain the generalized function with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 1, \alpha_4 = 2$, $\alpha_5 = 2$ corresponding to $n = 6-\epsilon$, i.e.

$$V_{\scriptscriptstyle 12}^{\scriptscriptstyle G} = -\omega^2\,V_{\scriptscriptstyle G}^{1,1|1,2|2}(n=6-\epsilon). \tag{203}$$

Under the same assumption we eliminate the integral with $P_{\scriptscriptstyle 21}$ in favor of the integral with $P_{\scriptscriptstyle 22}$ which contains a factor $1-y_1$ and obtain a combination of three generalized functions

$$V_{\scriptscriptstyle 22}^{\scriptscriptstyle G} = -\,\omega^2\left[V_{\scriptscriptstyle G}^{1,2|1,2|1} + V_{\scriptscriptstyle G}^{2,1|1,2|1} + V_{\scriptscriptstyle G}^{1,1|1,2|2}\right]\Big|_{n=6-\epsilon}. \tag{204}$$

9.4.2 Rank two tensor integrals in the $V^{\scriptscriptstyle G}$ family

Tensor integrals become ultraviolet divergent; with $q_{i\mu}\,q_{i\nu}$ in the numerator the integrals are overall divergent with (α,γ) or (β,γ) sub-diagram divergent. With $q_{1\mu}\,q_{2\nu}$ the function is overall divergent (but sub-diagrams are convergent).

Henceforth we want to analyze the tensor integrals with two powers of q_2 in the numerator: adopting the usual decomposition in form factors we have that

$$V^{\scriptscriptstyle G}(0|\mu,\nu\,;\,\cdots) = V_{\scriptscriptstyle 221}^{\scriptscriptstyle G}\,p_{1\mu}\,p_{1\nu} + V_{\scriptscriptstyle 222}^{\scriptscriptstyle G}\,p_{2\mu}\,p_{2\nu} + V_{\scriptscriptstyle 223}^{\scriptscriptstyle G}\,\{p_1p_2\}_{\mu\nu} + V_{\scriptscriptstyle 224}^{\scriptscriptstyle G}\,\delta_{\mu\nu}\,, \tag{205}$$

with the symmetrized product of Eq.(13). The integral representation for the form factors introduced in Eq.(205) is given by

$$V_{\scriptscriptstyle 22i}^{\scriptscriptstyle G} = -\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{\scriptscriptstyle G}\,R_{\scriptscriptstyle 22i;G}\,\chi_{\scriptscriptstyle G}^{-1-\epsilon}, \quad i \neq 4, \qquad V_{\scriptscriptstyle 224}^{\scriptscriptstyle G} = -\,\frac{1}{2}\,\Gamma\,(\epsilon)\int \mathcal{D}V_{\scriptscriptstyle G}\,\chi_{\scriptscriptstyle G}^{-\epsilon},$$
$$R_{\scriptscriptstyle 221;G} = \overline{Y}_2^2, \quad R_{\scriptscriptstyle 222;G} = (1-y_1)^2, \quad R_{\scriptscriptstyle 223;G} = (1-y_1)\,\overline{Y}_2, \tag{206}$$

where $X = (1-x_1)/(1-x_2)$ and $\overline{Y}_2 = 1-y_2\,\overline{X}$. The reduction of the form factors of the V^{G}_{22i} family proceeds as follows: at first we take the trace in both sides of Eq.(205) and obtain

$$p_1^2\,V^{G}_{221} + 2\,p_{12}\,V^{G}_{223} + p_2^2\,V^{G}_{222} + n\,V^{G}_{224} = -(p_1^2+m_4^2)V^{G}_{0}(p_1,p_1,P,\{m\}_{12345})$$
$$-\,2\,p_1^2\,V^{G}_{21}(p_1,p_1,P,\{m\}_{12345}) - 2\,p_{12}\,V^{G}_{22}(p_1,p_1,P,\{m\}_{12345}) + V^{E}_{0}(-p_2,-P,\{m\}_{5321}). \tag{207}$$

Similarly, contracting Eq.(205) with $p_{2\nu}$, we get additional relations

$$\begin{aligned}
p_{12}\,V^{G}_{221} + p_2^2\,V^{G}_{223} &= \frac{1}{2}\Big[(l_{154}-P^2)\,V^{G}_{21}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad +\,V^{E}_{11}(-p_2,-P,\{m\}_{5321}) -\,V^{E}_{11}(0,-p_1,\{m\}_{4321}) \\
&\quad +\,V^{E}_{0}(-p_2,-P,\{m\}_{5321}) -\,V^{E}_{0}(0,-p_1,\{m\}_{4321})\Big], \\
p_2^2\,V^{G}_{222} + p_{12}\,V^{G}_{223} + V^{G}_{224} &= \frac{1}{2}\Big[(l_{154}-P^2)\,V^{G}_{22}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad +\,V^{E}_{12}(-p_2,-P,\{m\}_{5321}) +\,V^{E}_{0}(0,-p_1,\{m\}_{4321})\Big].
\end{aligned} \tag{208}$$

In Eq.(208), where necessary, the momenta have been permuted to bring the integrand in the standard form of Eq.(119).

Eqs.(207)–(208) can be solved for the form factors with $i<4$ when we use one generalized scalar function,

$$V^{G}_{224} = \frac{1}{2}\,\omega^2\,V^{1,1|1,1|2}(n=6-\epsilon)\,. \tag{209}$$

The $q_{1\mu}\,q_{2\nu}$ tensor integral can be expressed in terms of form factors as follows:

$$V^{G}(\mu|\nu\,;\,\cdots) = V^{G}_{121}\,p_{1\mu}\,p_{1\nu} + V^{G}_{122}\,p_{2\mu}\,p_{2\nu} + V^{G}_{123}\,p_{1\mu}\,p_{2\nu} + V^{G}_{125}\,p_{1\nu}\,p_{2\mu} + V^{G}_{124}\,\delta_{\mu\nu}\,. \tag{210}$$

$V^{G}(\mu|\nu\,;\,\cdots)$ is not symmetric in μ and ν, and we have to distinguish between V^{G}_{123} and V^{G}_{125}. The integral representation for the form factors of Eq.(210) is:

$$\begin{aligned}
&V^{G}_{12i} = -\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_G\,R_{12i;G}\,\chi_G^{-1-\epsilon}, \quad i\neq 4, \qquad V^{G}_{124} = -\,\frac{1}{2}\,\Gamma\,(\epsilon)\int \mathcal{D}V_G\,x_2\,\chi_G^{-\epsilon}, \\
&R_{121;G} = \overline{Y}_2\,(1-x_1+x_2\,\overline{Y}_2)\,, \quad R_{122;G} = x_2\,(1-y_1)^2, \\
&R_{123;G} = (1-y_1)\,(1-x_1+x_2\,\overline{Y}_2), \quad R_{125;G} = (1-y_1)\,x_2\,\overline{Y}_2.
\end{aligned} \tag{211}$$

Since the q_1 sub-diagram involves three propagators, it is not possible to rewrite $V^{G}(\mu|\nu)$ in terms of $V^{G}(0|\mu,\nu)$. We can, however, express the five form factors of Eq.(210) in terms of scalar function employing the same technique adopted for the V^{G}_{22i} form factors. In fact, taking the trace of both sides of Eq.(210) one obtains the relation

$$p_1^2\,V^{G}_{121} + p_{12}\,(V^{G}_{123}+V^{G}_{125}) + p_2^2\,V^{G}_{122} + n\,V^{G}_{124} = \frac{1}{2}\Big[\,m_{31}^2\,V^{G}_{0}(p_1,p_1,P,\{m\}_{12345})$$
$$+\,V^{G}(0|\mu,\mu;p_1,p_1,P,\{m\}_{12345}) + V^{E}_{0}(0,p_2,\{m\}_{2345}) + B_0(p_1,\{m\}_{12})\,B_0(p_2,\{m\}_{45})\Big]; \tag{212}$$

contracting Eq.(210) with $p_{1\mu}$ we have

$$\begin{aligned}
p_1^2\,V^{G}_{121} + p_{12}\,V^{G}_{125} + V^{G}_{124} &= \frac{1}{2}\Big[-\,l_{112}\,V^{G}_{21}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad +\,V^{E}_{22}(p_1,P,\{m\}_{1345}) -\,V^{E}_{0}(0,p_2,\{m\}_{2345})\Big], \\
p_1^2\,V^{G}_{123} + p_{12}\,V^{G}_{122} &= \frac{1}{2}\Big[-\,l_{112}\,V^{G}_{22}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad +\,V^{E}_{21}(p_1,P,\{m\}_{1345}) -\,V^{E}_{21}(0,p_2,\{m\}_{2345})\Big];
\end{aligned} \tag{213}$$

finally, contracting Eq.(210) with $p_{2\nu}$ we have

$$
\begin{aligned}
p_2^2\,V^G_{123} + p_{12}\,V^G_{121} &= \frac{1}{2}\,\Big[(l_{145} - P^2)\,V^G_{11}(p_1,p_1,P,\{m\}_{12345}) - \,V^E_{11}(0,-p_1,\{m\}_{4321}) \\
&\quad + V^E_{12}(-p_2,-P,\{m\}_{5321}) - \,V^E_0(0,-p_1,\{m\}_{4321}) + \,V^E_0(-p_2,-P,\{m\}_{5321})\Big]\,, \\
p_2^2\,V^G_{122} + p_{12}V^G_{125} + V^G_{124} &= \frac{1}{2}\,\Big[(l_{145} - P^2)\,V^G_{12}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad + \,V^E_{22}(-p_2,-P,\{m\}_{5321}) + \,V^E_0(-p_2,-P,\{m\}_{5321})\Big]. \qquad (214)
\end{aligned}
$$

Furthermore we have

$$
V^G_{124} = \frac{1}{2}\,\omega^2\,V_G^{1,1|1,1|2}\,\Big|_{n=6-\epsilon}. \qquad (215)
$$

The system composed by Eqs.(212)–(214) gives the form factors with $i \neq 4$.

It is now necessary to analyze the form factors of the V^G_{11i} family, which are defined through the relation

$$
V^G(\mu,\nu|0\,;\,\cdots) = V^G_{111}\,p_{1\mu}\,p_{1\nu} + V^G_{112}\,p_{2\mu}\,p_{2\nu} + V^G_{113}\,\{p_1p_2\}_{\mu\nu} + V^G_{114}\,\delta_{\mu\nu}\,, \qquad (216)
$$

with $\{p_1p_2\}$ defined in Eq.(13). The integral representation for these form factors can be obtained with standard techniques:

$$
\begin{aligned}
V^G_{11i} &= -\Gamma\,(1+\epsilon)\int \mathcal{D}V_G\,R_{11i;G}\,\chi_G^{-1-\epsilon}, \qquad i \neq 4\,, \\
R_{111;G} &= (1 - x_1 + x_2\,\overline{Y})^2, \quad R_{112;G} = x_2^2\,(1-y_1)^2 \quad R_{113;G} = (1-y_1)\,x_2\,(1 - x_1 + x_2\,\overline{Y}_2), \\
V^G_{114} &= -\frac{1}{2}\,\Gamma\,(\epsilon)\int \mathcal{D}V_G\,\frac{x_2}{y_2}\,(1 - x_2 + x_2\,y_2)\,\chi_G^{-\epsilon}\,. \qquad (217)
\end{aligned}
$$

In order to reduce the form factors to linear combination of scalar functions, we start by taking the trace in both sides of Eq.(216) so that we obtain

$$
p_1^2\,V^G_{111} + 2\,p_{12}\,V^G_{113} + p_2^2\,V^G_{112} + n\,V^G_{114} = -m_1^2\,V^G_0(p_1,p_1,P,\{m\}_{12345}) + V^E_0(0,p_2,\{m\}_{2345}). \qquad (218)
$$

Contracting Eq.(216) with $p_{1\mu}$ we have additional relations,

$$
\begin{aligned}
p_1^2\,V^G_{111} + p_{12}V^G_{113} + V^G_{114} &= \frac{1}{2}\,\Big[-\,l_{112}\,V^G_{11}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad + \,V^E_{12}(p_1,P,\{m\}_{1345}) + \,V^E_0(0,p_2,\{m\}_{2345})\Big], \\
p_1^2\,V^G_{113} + p_{12}\,V^G_{112} &= \frac{1}{2}\,\Big[-\,l_{112}\,V^G_{12}(p_1,p_1,P,\{m\}_{12345}) \\
&\quad + \,V^E_{11}(p_1,P,\{m\}_{1345}) - \,V^E_{11}(0,p_2,\{m\}_{2345})\Big]. \qquad (219)
\end{aligned}
$$

It is not possible to obtain more equations by multiplying both sides of Eq.(210) by $p_{2\mu}$, since the scalar product $q_1 \cdot p_2$ is irreducible. Eqs.(218)–(219) give the form factors with $i \neq 2$ when we introduce one generalized scalar function,

$$
V^G_{112} = 4\,\omega^4\,V_G^{1,1|1,3|3}(n = 8-\epsilon)\,. \qquad (220)
$$

There are other equivalent solutions. Results for this family are summarized in Appendix B.4. $V^G_0 \equiv V^{221}_0$ is discussed in Sect. 7.1 of III (see comment at the end of Section 9.1.2), evaluation of form factors in Section 11.3.

9.5 The V^K-family ($\alpha = 2, \beta = 3, \gamma = 1$)

Next we consider the scalar diagram in the V^K-family of Fig. 10, which is overall ultraviolet convergent (with all sub-diagrams convergent) and which is representable as

$$\pi^4 V_0^K(P, p_1, P, \{m\}_{123456}) = \mu^{2\epsilon} \int d^n q_1 \int d^n q_2 \frac{1}{[1][2]_K[3]_K[4]_K[5]_K[6]_K}, \tag{221}$$

with propagators

$$[1] \equiv q_1^2 + m_1^2, \quad [2]_K \equiv (q_1 + P)^2 + m_2^2, \quad [3]_K \equiv (q_1 - q_2)^2 + m_3^2,$$
$$[4]_K \equiv q_2^2 + m_4^2, \quad [5]_K \equiv (q_2 + p_1)^2 + m_5^2, \quad [6]_K \equiv (q_2 + P)^2 + m_6^2. \tag{222}$$

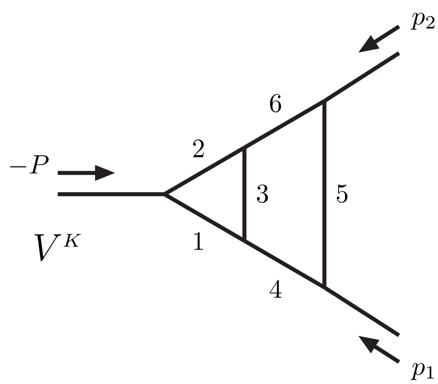


Figure 10: The irreducible two-loop vertex diagrams V^K. External momenta flow inwards. Internal masses are enumerated according to Eq.(222).

9.5.1 *Vector integrals in the V^K family*

The form factors for the vector integrals are defined by

$$V^K(\mu\,|\,0\,;\, P, p_1, P, \{m\}_{123456}) = \sum_{i=1}^{2} V_{1i}^K(P, p_1, P, \{m\}_{123456})\, p_{i\mu},$$
$$V^K(0\,|\,\mu\,;\, P, p_1, P, \{m\}_{123456}) = \sum_{i=1}^{2} V_{2i}^K(P, p_1, P, \{m\}_{123456})\, p_{i\mu}, \tag{223}$$

where the form factors $V_{ij}^K(k, p, k, \{m\}_{a\cdots f})$ refer to the basis p_μ and $(k-p)_\mu$. Their explicit expression is

$$V_{ij}^K = -\,\Gamma\,(2+\epsilon) \int \mathcal{D}V_K\, P_{ij;K}\, \chi_K^{-2-\epsilon},$$
$$\int \mathcal{D}V_K = \omega^\epsilon \int dS_2(\{x\}) \int dS_3(\{y\}) \Big[x_2\,(1-x_2)\Big]^{-1-\epsilon/2} y_3^{\epsilon/2},$$
$$P_{00;K} = 1, \;\; P_{11;K} = -\,H_2, \;\; P_{12;K} = -\,H_1, \;\; P_{21;K} = Y_2, \;\; P_{22;K} = Y_1, \tag{224}$$

where ω is defined in Eq.(15), and the quantities Y_i and H_i are given in Eq.(11). The polynomial χ_K is given by

$$\chi_K = -\,F(y_2 - X\,y_3\,,\, y_1 - X\,y_3) + l_{265}\,y_1 + (P^2 - l_{245})\,y_2 - (2\,X\,P^2 - m_{xx}^2 + m_4^2)\,y_3 + m_6^2, \tag{225}$$

with F defined in Eq.(12) and

$$m_{xx}^2 = \frac{-P^2 x_1^2 + x_1(P^2 + m_{12}^2) + x_2 m_{312}^2}{x_2(1-x_2)},$$

with $X = (1-x_1)/(1-x_2)$. The generalized function in this family is

$$\begin{aligned}
&V_{\scriptscriptstyle K}^{\alpha_1,\alpha_2|\alpha_4,\alpha_5,\alpha_6|\alpha_3}(n = \sum_{i=1}^{6} \alpha_i - 2 - \epsilon) = \pi^{-4}\,(\mu^2)^{4-n} \int d^n q_1\, d^n q_2 \prod_{i=1}^{6} [i]_{\scriptscriptstyle K}^{-\alpha_i} \\
&= -\frac{\Gamma(2+\epsilon)}{\prod_{i=1}^{6}\Gamma(\alpha_i)}\,\omega^{6-\sum_{j=1}^{6}\alpha_j+\epsilon} \int dS_2(\{x\}) \int dS_3(\{y\})\,(1-x_1)^{\rho_1} \\
&\times\; (x_1-x_2)^{\rho_2}\, x_2^{\rho_3}\,(1-x_2)^{\rho_4}\,(1-y_1)^{\rho_5}\,(y_1-y_2)^{\rho_6}\,(y_2-y_3)^{\rho_7}\, y_3^{\rho_8}\,\chi_{\scriptscriptstyle K}^{-2-\epsilon},
\end{aligned} \tag{226}$$

where $[1]_{\scriptscriptstyle K} \equiv [1]$, ω is defined in Eq.(15) and the powers ρ_i are

$$\begin{aligned}
&\rho_1 = \alpha_2 - 1, \quad \rho_2 = \alpha_1 - 1, \quad \rho_3 = \frac{1}{2}\,(\sum\alpha - 2\,\alpha_1 - 2\,\alpha_2 - 4 - \epsilon), \\
&\rho_4 = \frac{1}{2}\,(\sum\alpha - 2\,\alpha_1 - 2\,\alpha_2 - 2\,\alpha_3 - 2 - \epsilon), \quad \rho_5 = \alpha_6 - 1, \quad \rho_6 = \alpha_5 - 1, \\
&\rho_7 = \alpha_4 - 1, \quad \rho_8 = \frac{1}{2}\,(\epsilon - \sum\alpha + 2\,\alpha_1 + 2\,\alpha_2 + 2\,\alpha_3).
\end{aligned} \tag{227}$$

There is partial reducibility with respect to q_1 and complete reducibility with respect to q_2. We obtain

$$\begin{aligned}
V^{\scriptscriptstyle K}(P|0;P,p_1,P,\{m\}_{123456}) &= -\frac{1}{2}\Big[l_{\scriptscriptstyle P12}V_{\scriptscriptstyle 0}^{\scriptscriptstyle K}(P,p_1,P,\{m\}_{123456}) - V_{\scriptscriptstyle 0}^{\scriptscriptstyle I}(p_1,P,\{m\}_{13456}) \\
&\quad + V_{\scriptscriptstyle 0}^{\scriptscriptstyle I}(-p_2,-P,\{m\}_{23654})\Big], \\
V^{\scriptscriptstyle K}(0|p_1;P,p_1,P,\{m\}_{123456}) &= -\frac{1}{2}\Big[l_{145}V_{\scriptscriptstyle 0}^{\scriptscriptstyle K}(P,p_1,P,\{m\}_{123456}) + V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(P,P,p_1,\{m\}_{12365}) \\
&\quad - V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(P,P,0,\{m\}_{12364})\Big], \\
V^{\scriptscriptstyle K}(0|P;P,p_1,P,\{m\}_{123456}) &= -\frac{1}{2}\Big[l_{\scriptscriptstyle P46}V_{\scriptscriptstyle 0}^{\scriptscriptstyle K}(P,p_1,P,\{m\}_{123456}) + V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(P,P,p_1,\{m\}_{12365}) \\
&\quad - V_{\scriptscriptstyle 0}^{\scriptscriptstyle G}(-P,-P,-p_2,\{m\}_{21345})\Big].
\end{aligned} \tag{228}$$

We can write

$$V^{\scriptscriptstyle K}(P\,|\,0\,;\,P,p_1,P,\{m\}_{123456}) = p_1\cdot P\, V_{\scriptscriptstyle 11}^{\scriptscriptstyle K} + p_2\cdot P\,\Big[V_{\scriptscriptstyle 11}^{\scriptscriptstyle K} - I_{\scriptscriptstyle R;K}\Big], \tag{229}$$

$$I_{\scriptscriptstyle R;K} = \Gamma(2+\epsilon)\int \mathcal{D}V_{\scriptscriptstyle K}\, x_2\,(y_1-y_2)\,\chi_{\scriptscriptstyle K}^{-2-\epsilon} = \omega^2\, V_{\scriptscriptstyle K}^{1,1|1,2,1|2}(n = 6-\epsilon), \tag{230}$$

which gives the reduction of the 11-component. Reduction of the 12-component follows from

$$V_{\scriptscriptstyle 12}^{\scriptscriptstyle K} = V_{\scriptscriptstyle 11}^{\scriptscriptstyle K} - I_{\scriptscriptstyle R;K}. \tag{231}$$

A similar argument holds for the $2i$ components with the same $I_{\scriptscriptstyle R;K}$, although the reduction of the $V_{\scriptscriptstyle 2i}^{\scriptscriptstyle K}$ components can also be obtained solving the system composed by the last two equations of Eq.(228).

9.5.2 Rank two tensor integrals in the $V^{\scriptscriptstyle K}$ family

Only the $q_{1\mu}\,q_{1\nu}$ tensor integral has an ultraviolet divergent (α,γ) sub-diagram.

Henceforth we consider the $q_{2\mu}\, q_{2\nu}$ tensor integral: we introduce the form factors of the V^{K}_{22i} family through the relation

$$V^{K}(0|\mu,\nu\,;\,\cdots) = V^{K}_{221}\, p_{1\mu}\, p_{1\nu} + V^{K}_{222}\, p_{2\mu}\, p_{2\nu} + V^{K}_{223}\, \{p_1\, p_2\}_{\mu\nu} + V^{K}_{224}\, \delta_{\mu\nu}\,, \tag{232}$$

with the symmetrized product of Eq.(13). Their integral representation is given by

$$V^{K}_{22i} = -\Gamma\,(2+\epsilon)\int \mathcal{D}V_K\, R_{22i;K}\, \chi_K^{-2-\epsilon}, \quad i \neq 4, \qquad V^{K}_{224} = -\frac{1}{2}\Gamma\,(1+\epsilon)\int \mathcal{D}V_K\, \chi_K^{-1-\epsilon},$$

$$R_{221;K} = Y_2^2, \qquad R_{222;K} = Y_1^2, \qquad R_{223;K} = Y_1\, Y_2. \tag{233}$$

We want to express the V^{K}_{22i} form factors as linear combinations of scalar functions. Taking the trace of both sides of Eq.(232) one obtains

$$p_1^2\, V^{K}_{221} + 2\, p_{12}\, V^{K}_{223} + p_2^2\, V^{K}_{222} + n\, V^{K}_{224} = V^{G}_{0}(P,P,p_1,\{m\}_{12365}) - m_4^2\, V^{K}_{0}(P,p_1,P,\{m\}_{123456}). \tag{234}$$

Contracting both sides of Eq.(232) with $p_{2\mu}$ we get

$$\begin{aligned}
p_{12}\, V^{K}_{221} + p_2^2\, V^{K}_{223} = \frac{1}{2}\Big[&(l_{165} - P^2)\, V^{K}_{21}(P,p_1,P,\{m\}_{123456}) - V^{G}_{21}(P,P,0,\{m\}_{12364})\\
&- V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\\
&+ V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
p_2^2\, V^{K}_{222} + p_{12}\, V^{K}_{223} + V^{K}_{224} = \frac{1}{2}\Big[&(l_{165} - P^2)\, V^{K}_{22}(p_1,p_1,P,\{m\}_{123456}) - V^{G}_{21}(P,P,0,\{m\}_{12364})\\
&- V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\\
&- V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big].
\end{aligned} \tag{235}$$

Once again, one should be particularly careful in shifting the integration momenta in order to bring the integrand of the V^G functions in the chosen standard form:

$$\frac{\mu^{2\epsilon}}{\pi^4}\int d^n r_1 d^n r_2 \frac{r_{i\mu}}{D_a D_b D_c D_d D_e} \equiv V^{G}_{i1}(k_b,k_b,k_e,\{m\}_{abcde})\, k_{b\mu} + V^{G}_{i2}(k_b,k_b,k_e,\{m\}_{abcde})\, (k_e - k_b)_\mu \tag{236}$$

$$\begin{aligned}
&D_a = r_1^2 + m_a^2\,, && D_b = (r_1+k_b)^2 + m_b^2\,, && D_c = (r_2 - r_2)^2 + m_c^2\,,\\
&D_d = (r_1+k_b)^2 + m_d^2\,, && D_e = (r_2+k_e)^2 + m_e^2\,.
\end{aligned}$$

Contracting both sides of Eq.(232) with $p_{1\mu}$ we get

$$\begin{aligned}
p_{12}\, V^{K}_{222} + p_1^2\, V^{K}_{223} = \frac{1}{2}\Big[&- l_{145}\, V^{K}_{22}(P,p_1,P,\{m\}_{123456}) + V^{G}_{21}(P,P,0,\{m\}_{12364})\\
&- V^{G}_{22}(P,P,0,\{m\}_{12364})\Big],\\
p_1^2\, V^{K}_{221} + p_{12}\, V^{K}_{223} + V^{K}_{224} = \frac{1}{2}\Big[&- l_{145}\, V^{K}_{21}(P,p_1,P,\{m\}_{123456}) - V^{G}_{21}(P,P,p_1,\{m\}_{12365})\\
&+ V^{G}_{21}(P,P,0,\{m\}_{12364}) - V^{G}_{22}(P,P,0,\{m\}_{12364})\Big].
\end{aligned} \tag{237}$$

A solution of Eqs.(234)–(237) give the form factors in the 22 group. We can now analyze the $q_{1\mu}\, q_{2\nu}$ tensor integrals. As for the V^G case, this tensor integral is not symmetric in $\mu\nu$, so that we need to introduce five form factors:

$$V^{K}(\mu|\nu\,;\,\cdots) = V^{K}_{121}\, p_{1\mu}\, p_{1\nu} + V^{K}_{122}\, p_{2\mu}\, p_{2\nu} + V^{K}_{123}\, p_{1\mu}\, p_{2\nu} + V^{K}_{125}\, p_{1\nu}\, p_{2\mu} + V^{K}_{124}\, \delta_{\mu\nu}. \tag{238}$$

The integral representation of these form factors is the following:

$$V^{K}_{12i} = -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_K\, R_{12i;K}\,\chi_K^{-2-\epsilon}, \quad i\neq 4, \qquad V^{K}_{124} = -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_K\, x_2\,\chi_K^{-1-\epsilon},$$
$$R_{121;K} = -Y_2\,H_2, \qquad R_{122;K} = -Y_1\,H_1\,, \qquad R_{123;K} = -Y_1\,H_2\,, \qquad R_{125;K} = -Y_2\,H_1. \tag{239}$$

Employing the usual procedure we can reduce the form factors. Contracting Eq.(238) with $\delta_{\mu\nu}$, $p_{1\nu}$ and $p_{2\nu}$ we obtain

$$p_1^2\,V^{K}_{121} + p_{12}\,(V^{K}_{123}+V^{K}_{125}) + p_2^2\,V^{K}_{122} + n\,V^{K}_{124} = -\,\frac{1}{2}\Big[\,m^2_{134}\,V^{K}_{0}(P,p_1,P,\{m\}_{123456})$$
$$-\,V^{G}_{0}(P,P,p_1,\{m\}_{12365}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654}) - B_0(P,\{m\}_{12})\,C_0(p_1,p_2,\{m\}_{456})\Big], \tag{240}$$

$$\begin{aligned}
p_1^2\,V^{K}_{121} + p_{12}\,V^{K}_{123} + V^{K}_{124} &= \frac{1}{2}\Big[-l_{145}\,V^{K}_{11}(P,p_1,P,\{m\}_{123456}) + V^{G}_{11}(P,P,0,\{m\}_{12364})\\
&\quad -\,V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{G}_{12}(P,P,0,\{m\}_{12364})\Big],\\
p_1^2\,V^{K}_{125} + p_{12}\,V^{K}_{122} &= \frac{1}{2}\Big[-l_{145}\,V^{K}_{12}(P,p_1,P,\{m\}_{123456}) + V^{G}_{11}(P,P,0,\{m\}_{12364})\\
&\quad -\,V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{G}_{12}(P,P,0,\{m\}_{12364})\\
&\quad +\,V^{G}_{12}(P,P,p_1,\{m\}_{12365})\Big],
\end{aligned} \tag{241}$$

$$\begin{aligned}
p_2^2\,V^{K}_{122} + p_{12}V^{K}_{125} + V^{K}_{124} &= \frac{1}{2}\Big[(l_{165}-P^2)\,V^{K}_{12}(P,p_1,P,\{m\}_{123456}) + V^{G}_{12}(P,P,0,\{m\}_{12364})\\
&\quad -\,V^{G}_{11}(P,P,0,\{m\}_{12364}) - V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad -\,V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
p_2^2\,V^{K}_{123} + p_{12}\,V^{K}_{121} &= \frac{1}{2}\Big[(l_{165}-P^2)\,V^{K}_{11}(P,p_1,P,\{m\}_{123456}) - V^{G}_{11}(P,P,0,\{m\}_{12364})\\
&\quad -\,V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{12}(P,P,0,\{m\}_{12364})\\
&\quad +\,V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big].
\end{aligned} \tag{242}$$

The solution of Eqs.(240)–(242) gives the form factors in the 12 group.

Finally, we consider the $q_{1\mu}\,q_{1\nu}$ tensor integral for which we introduce the form factors V^{K}_{11i}:

$$V^{K}(\mu,\nu|0\,;\,\cdots) = V^{K}_{111}\,p_{1\mu}\,p_{1\nu} + V^{K}_{112}\,p_{2\mu}\,p_{2\nu} + V^{K}_{113}\,\{p_1p_2\}_{\mu\nu} + V^{K}_{114}\,\delta_{\mu\nu}\,, \tag{243}$$

where the symmetrized product is given in Eq.(13). Their integral representation is given by

$$\begin{aligned}
V^{K}_{11i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_K\, R_{11i;K}\,\chi_K^{-2-\epsilon}, \qquad i\neq 4\,,\\
R_{111;K} &= H_2^2\,, \quad R_{112;K} = H_1^2\,, \quad R_{113;K} = H_1\,H_2,\\
V^{K}_{114} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_K\,\Big[\frac{x_2\,(1-x_2)}{y_3} + x_2^2\Big]\,\chi_K^{-1-\epsilon}\,.
\end{aligned} \tag{244}$$

H_1 and H_2 were defined in Eq.(11). Contracting both sides of Eq.(243) first with $\delta_{\mu\nu}$ and then with P_ν, it is possible to obtain the following set of three equations:

$$p_1^2\,V^{K}_{111} + 2\,p_{12}\,V^{K}_{113} + p_2^2\,V^{K}_{122} + n\,V^{K}_{124} = -m_1^2\,V^{K}_{0}(P,p_1,P,\{m\}_{123456}) + V^{I}_{0}(-p_2,-P,\{m\}_{23654}), \tag{245}$$

$$p_1 \cdot P\, V^{K}_{111} + p_2 \cdot P V^{K}_{113} + V^{K}_{114} = \frac{1}{2}\Big[- l_{P12}\, V^{K}_{11}(P, p_1, P, \{m\}_{123456}) + V^{I}_{11}(p_1, P, \{m\}_{13456}) + V^{I}_{12}(-P, -p_2, \{m\}_{23645}) + V^{I}_{0}(-P, -p_2, \{m\}_{23645})\Big],$$

$$p_1 \cdot P\, V^{K}_{113} + p_2 \cdot P\, V^{K}_{112} + V^{K}_{114} = \frac{1}{2}\Big[- l_{P12}\, V^{K}_{12}(P, p_1, P, \{m\}_{123456}) + V^{I}_{12}(p_1, P, \{m\}_{13456}) + V^{I}_{11}(-P, -p_2, \{m\}_{23645}) + V^{I}_{0}(-P, -p_2, \{m\}_{23645})\Big]. \quad (246)$$

We have then three equations and four unknown form factors, so that we should look for relations between form factors and generalized scalar function; for example we have that

$$V^{K}_{111} - 2V^{K}_{113} + V^{K}_{112} = -\Gamma(2+\epsilon)\int \mathcal{D}V_{K}\, x_2^2 (y_1 - y_2)^2\, \chi_{K}^{-2-\epsilon} = 4\omega^4 V_{K}^{1,1|1,3,1|3}\Big|_{n=8-\epsilon}. \quad (247)$$

Results for this family are summarized in Appendix B.5. $V^{K}_{0} \equiv V^{231}_{0}$ is discussed in Sects. 9.1 - 9.2 of III (see comment at the end of Section 9.1.2), evaluation of form factors in Section 11.4.

9.6 The V^{H}-family ($\alpha = 2, \beta = 2, \gamma = 2$)

Finally, we consider the non-planar diagram of the V^{H}-family, given in Fig. 11, which is representable as

$$\pi^4\, V^{H}_{0}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) = \mu^{2\epsilon} \int d^n q_1 \int d^n q_2\, \frac{1}{[1][2]_H [3]_H [4]_H [5]_H [6]_H}, \quad (248)$$

with propagators

$$[1] \equiv q_1^2 + m_1^2, \quad [2]_H \equiv (q_1 - p_2)^2 + m_2^2, \quad [3]_H \equiv (q_1 - q_2 + p_1)^2 + m_3^2,$$

$$[4]_H \equiv (q_1 - q_2 - p_2)^2 + m_4^2, \quad [5]_H \equiv q_2^2 + m_5^2, \quad [6]_H \equiv (q_2 - p_1)^2 + m_6^2. \quad (249)$$

The basis for the form factors $V^{H}_{i\cdots j}(k, p, k, -p, \{m\}_{a\cdots f})$ is chosen to be p_μ and $-k_\mu$. All members of this

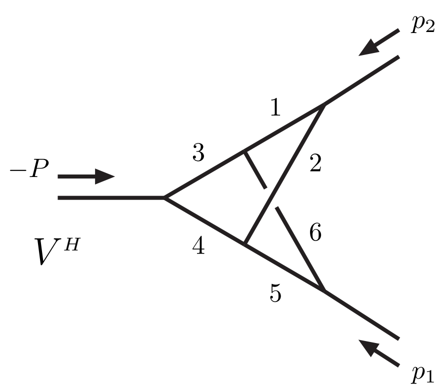


Figure 11: The irreducible two-loop vertex diagrams V^{H}. External momenta flow inwards. Internal masses are enumerated according to Eq.(249).

family, including rank-two tensors, are overall ultraviolet convergent with all sub-diagrams convergent.

Adopting the parametrization presented in Sect. 10.2 of III, the integral representation for the scalar integral of the V^{H} family, with arbitrary powers for the propagators, is

$$V_{H}^{\alpha_1,\alpha_2|\alpha_5,\alpha_6|\alpha_3,\alpha_4}(n = \sum_{i=1}^{6}\alpha_i - 2 - \epsilon) = \pi^{-4}\,(\mu^2)^{4-n} \int d^n q_1\, d^n q_2 \prod_{i=1}^{6} [i]_{H}^{-\alpha_i}$$

$$= -\frac{\Gamma(2+\epsilon)}{\prod_{i=1}^{6}\Gamma(\alpha_i)}\, \omega^{6-\sum_{j=1}^{6}\alpha_j+\epsilon} \int dC_2(x,y) \int dC_3(\{z\})$$

$$\times\, (1-z_1)^{\rho_1}\, z_1^{\rho_2}\, (1-z_2)^{\rho_3}\, z_2^{\rho_4}\, (1-z_3)^{\rho_5}\, z_3^{\rho_6}\, (1-y)^{\rho_7}\, y^{\rho_8}\, (1-x)^{\rho_9}\, x^{\rho_{10}}\, \chi_{H}^{-2-\epsilon}, \quad (250)$$

where $[1]_H \equiv [1]$, where ω is defined in Eq.(15) and the powers ρ_i $(i = 1, \dots, 10)$ are

$$\begin{aligned}
&\rho_1 = \alpha_1 - 1, \quad \rho_2 = \alpha_2 - 1, \quad \rho_3 = \alpha_4 - 1, \quad \rho_4 = \alpha_3 - 1,\\
&\rho_5 = \alpha_5 - 1, \quad \rho_6 = \alpha_6 - 1, \quad \rho_7 = \alpha_5 + \alpha_6 - 1, \quad \rho_8 = \frac{1}{2}(\epsilon - \sum \alpha + 2\,\alpha_1 + 2\,\alpha_2 + 2\,\alpha_3 + 2\,\alpha_4),\\
&\rho_9 = -\frac{1}{2}(4 + \epsilon - \sum \alpha + 2\,\alpha_1 + 2\,\alpha_2), \quad \rho_{10} = -\frac{1}{2}\,(4 + \epsilon - \sum \alpha + 2\,\alpha_3 + 2\,\alpha_4).
\end{aligned} \tag{251}$$

The polynomial χ_H is given by $\chi_H = -Q^2\,y^2 + (M_x^2 - M^2 + Q^2)\,y + M^2$ where:

$$Q_\mu = K_{1\mu} - K_{2\mu} - K_{3\mu}, \qquad M^2 = R_3^2 - K_3^2, \qquad M_x^2 = \frac{x\,(R_1^2 - K_1^2) + (1-x)\,(R_2^2 - K_2^2)}{x\,(1-x)}\,,$$

$$\begin{aligned}
&R_1^2 = l_{212}\,z_1 + m_1^2, \quad R_2^2 = z_2\,(p_1^2 + m_3^2) + (1 - z_2)\,(p_2^2 + m_4^2), \quad R_3^2 = l_{156}\,z_3 + m_5^2,\\
&K_{1\mu} = -z_1\,p_{2\mu}, \quad K_{2\mu} = z_2\,p_{1\mu} - (1 - z_2)\,p_{2\mu}, \quad K_{3\mu} = -z_3\,p_{1\mu}.
\end{aligned}$$

9.6.1 Vector integrals in the V^H family

The form factors for the vector integrals are defined by the relations

$$\begin{aligned}
V^H(\mu\,|\,0\,;\,-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) &= \sum_{i=1}^{2} V^H_{1i}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456})\,p_{i\mu},\\
V^H(0\,|\,\mu\,;\,-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) &= \sum_{i=1}^{2} V^H_{2i}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456})\,p_{i\mu}.
\end{aligned} \tag{252}$$

Their explicit expression, in terms of integrals over the Feynman parameters, is

$$\begin{aligned}
&V^H_{ij} = -\,\Gamma\,(2 + \epsilon) \int \mathcal{D}V_H\, P_{ij;H}\, \chi_H^{-2-\epsilon},\\
&\int \mathcal{D}V_H = \omega^\epsilon \int dC_5(x, y, \{z\}) \Big[x\,(1-x)\Big]^{-1-\epsilon/2} y^{1+\epsilon/2}\,(1-y),\\
&P_{00;H} = 1, \quad P_{11;H} = -(z_2 - z_3)\,(1-x)\,(1-y), \quad P_{12;H} = (1 - z_1 - z_2)\,(1-x)\,(1-y),\\
&P_{21;H} = y\,(z_2 - z_3), \quad P_{22;H} = -y\,(1 - z_1 - z_2),
\end{aligned} \tag{253}$$

where ω is defined in Eq.(15) and P_{00} is the factor that arises in the calculation of the scalar integral. The form factors for the vector integrals can be reduced as follows: first it is possible to simplify the scalar products $q_1 \cdot p_2$ and $q_2 \cdot p_1$, respectively, obtaining the relations

$$\begin{aligned}
V^H(p_2\,|\,0\,;\,-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) = \frac{1}{2}\Big[&l_{212}\,V^H_0(-p_2, p_1, -p_2, -p_1, \{m\}_{123456})\\
&-\,V^G_0(p_1, p_1, -p_2, \{m\}_{56134})\\
&+\,V^G_0(-P, -P, -p_2, \{m\}_{34256})\Big],\\
V^H(0\,|\,p_1\,;\,-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) = \frac{1}{2}\Big[&l_{156}\,V^H_0(-p_2, p_1, -p_2, -p_1, \{m\}_{123456})\\
&+\,V^G_0(p_2, p_2, -p_1, \{m\}_{21634})\\
&-\,V^G_0(p_2, p_2, -p_1, \{m\}_{12543})\Big].
\end{aligned} \tag{254}$$

Since there are no other reducible scalar products we must find relations that link the form factors of the vector integrals to a linear combination of generalized scalar functions. The following identities hold:

$$\begin{aligned}
V^H_{22}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) &= \omega^2 \Big[V_H^{1,2|1,1|2,1} - V_H^{2+1|1,1|1,2}\Big]\Big|_{n=6-\epsilon},\\
V^H_{11}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) &= \omega^2 \Big[V_H^{1,1|1,2|1,2} - V_H^{1,1|2,1|2,1}\Big]\Big|_{n=6-\epsilon}.
\end{aligned} \tag{255}$$

9.6.2 Rank two tensor integrals in the V^{H} family

It is then necessary to consider the tensor integrals that have two momenta of integration with free Lorentz indices. We start from the the $V^{H}(0|\mu,\nu)$ integral and introduce the relevant form factors through the relation

$$V^{H}(0|\mu,\nu\,;\,\cdots) = V^{H}_{221}\,p_{1\mu}\,p_{1\nu} + V^{H}_{222}\,p_{2\mu}\,p_{2\nu} + V^{H}_{223}\,\{p_1p_2\}_{\mu\nu} + V^{H}_{224}\,\delta_{\mu\nu}. \tag{256}$$

The integral representation of these form factors is given by

$$\begin{aligned}
V^{H}_{22i} &= -\,\Gamma\,(2+\epsilon)\int\mathcal{D}V_{H}\,y^2\,R_{22i;H}\,\chi_{H}^{-2-\epsilon}, \quad i\neq 4, \qquad V^{H}_{224} = -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int\mathcal{D}V_{H}\,\chi_{H}^{-1-\epsilon},\\
R_{221;H} &= (z_2-z_3)^2, \quad R_{222;H} = (1-z_1-z_2)^2\,, \quad R_{223;H} = -\,(1-z_1-z_2)\,(z_2-z_3)\,.
\end{aligned} \tag{257}$$

Multiplying Eq.(256) by $\delta_{\mu\nu}$ and $p_{1\mu}$, respectively, we obtain the relations

$$\begin{aligned}
p_1^2V^{H}_{221}+p_2^2V^{H}_{222}+2p_{12}V^{H}_{223}+nV^{H}_{224} &= V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) - m_5^2\,V^{H}_{0}(-p_2,p_1,-p_2,-p_1,\{m\}_{123456})\,,\\
p_1^2\,V^{H}_{221} + p_{12}\,V^{H}_{223} + V^{H}_{224} &= \frac{1}{2}\Big[l_{156}\,V^{H}_{21}(-p_2,p_1,-p_2,-p_1,\{m\}_{123456}) - V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad -\; V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad +\; V^{G}_{22}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634})\Big],\\
p_1^2\,V^{H}_{223} + p_{12}\,V^{H}_{222} &= \frac{1}{2}\Big[l_{156}\,V^{H}_{22}(-p_2,p_1,-p_2,-p_1,\{m\}_{123456}) + V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad +\; V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad -\; V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad -\; V^{G}_{21}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634})\\
&\quad +\; V^{G}_{22}(-p_2,-p_2,p_1,\{m\}_{12543})\Big].
\end{aligned} \tag{258}$$

We then have a set of three equations that can be solved for $i\neq 4$ when we express one of the form factors in terms of a generalized scalar function; for example we have that

$$V^{H}_{224} = \frac{\omega^2}{2}\,\Big[V_{H}^{1,1|1,1|1,2} + V_{H}^{1,1|1,1|2,1} + V_{H}^{2,1|1,1|1,1} + V_{H}^{1,2|1,1|1,1}\Big]\Big|_{n=6-\epsilon}. \tag{259}$$

We can proceed in a completely analogous way for the $q_{1\mu}\,q_{1\nu}$ tensor integral. The relevant form factors are defined through the relation

$$V^{H}(\mu,\nu\,|0;\,\cdots) = V^{H}_{111}\,p_{1\mu}\,p_{1\nu} + V^{H}_{112}\,p_{2\mu}\,p_{2\nu} + V^{H}_{113}\,\{p_1p_2\}_{\mu\nu} + V^{H}_{114}\,\delta_{\mu\nu}. \tag{260}$$

The integral representation of these form factors is given by

$$\begin{aligned}
V^{H}_{11i} &= -\,\Gamma\,(2+\epsilon)\int\mathcal{D}V_{H}\,(1-x)^2\,(1-y)^2\,R_{11i;H}\,\chi_{H}^{-2-\epsilon}, \quad i\neq 4,\\
V^{H}_{114} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int\mathcal{D}V_{H}\,R_{114;H}\,\chi_{H}^{-1-\epsilon},\\
R_{11i;H} &= R_{22i;H}, \qquad\quad R_{114;H} = (1-x)\,(1-x+\frac{x}{y})\,.
\end{aligned} \tag{261}$$

Contracting Eq.(260) by $\delta_{\mu\nu}$ and $p_{2\mu}$, respectively, we obtain the relations

$$\begin{aligned}
p_1^2\,V^{H}_{111} + p_2^2\,V^{H}_{112} + 2\,p_{12}\,V^{H}_{113} + n\,V^{H}_{114} &= V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256})\\
&\quad -\,m_1^2\,V^{H}_{0}(-p_2,p_1,-p_2,-p_1,\{m\}_{123456})\,,
\end{aligned} \tag{262}$$

$$
\begin{aligned}
p_2^2\, V^{H}_{112} + p_{12}\, V^{H}_{113} + V^{H}_{114} &= \frac{1}{2}\Big[l_{212}\, V^{H}_{12}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) -\ V^{G}_{11}(-P, -P, -p_2, \{m\}_{34256}) \\
&+\ V^{G}_{21}(-P, -P, -p_2, \{m\}_{34256}) +\ V^{G}_{0}(-P, -P, -p_2, \{m\}_{34256}) \\
&+\ V^{G}_{22}(p_1, p_1, -p_2, \{m\}_{56234}) -\ V^{G}_{12}(p_1, p_1, -p_2, \{m\}_{56234})\Big], \\
p_2^2\, V^{H}_{113} + p_{12}\, V^{H}_{111} &= \frac{1}{2}\Big[l_{212}\, V^{H}_{11}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) +\ V^{G}_{11}(p_1, p_1, -p_2, \{m\}_{56234}) \\
&-\ V^{G}_{21}(p_1, p_1, -p_2, \{m\}_{56234}) +\ V^{G}_{22}(p_1, p_1, -p_2, \{m\}_{56234}) \\
&-\ V^{G}_{12}(p_1, p_1, -p_2, \{m\}_{56234})\Big]. \qquad (263)
\end{aligned}
$$

Once again, we can rewrite one of the form factors of the V^{H}_{11i} family as a linear combination of generalized scalar functions and solve the system for the others: or instance

$$
\begin{aligned}
V^{H}_{111} &= 4\,\omega^4\left[V_{H}^{1,1|3,1|3,1} + V_{H}^{1,1|1,3|1,3} - \frac{1}{2}\,V_{H}^{1,1|2,2|2,2}\right]\Big|_{n=8-\epsilon} \quad \text{or} \\
V^{K}_{114} &= \frac{1}{2}\,\omega^2\left[V_{H}^{1,1|1,2|1,1} + V_{H}^{1,1|2,1|1,1} + V_{H}^{1,1|1,1|1,2} + V_{H}^{1,1|1,1|2,1}\right]\Big|_{n=6-\epsilon}. \qquad (264)
\end{aligned}
$$

The $q_{1\mu}\, q_{2\nu}$ tensor integrals are symmetric in the exchange of μ and ν; this fact can be understood noticing that the integral with respect to q_1 is proportional to $a_1\, q_2^{\mu} + a_2\, Q^{\mu}$, where a_1 and a_2 are scalar factors and Q^{μ} is a linear combination of the external momenta. Therefore, after the q_1 integration, the integrand will split into a part proportional to $q_2^{\mu} q_2^{\nu}$, obviously symmetric with respect to $\mu \leftrightarrow \nu$, and into a part proportional to $q_2^{\nu} Q^{\mu}$; also the latter is symmetric since the vector integral with q_2^{ν} in the numerator is proportional to Q^{ν}.

To describe their tensor structure it is necessary to introduce four form factors:

$$
V^{H}(\mu|\nu;\cdots) = V^{H}_{121}\, p_{1\mu}\, p_{1\nu} + V^{H}_{122}\, p_{2\mu}\, p_{2\nu} + V^{H}_{123}\, \{p_1\, p_2\}_{\mu\nu} + V^{H}_{124}\, \delta_{\mu\nu}, \qquad (265)
$$

with $\{p_1 p_2\}$ given in Eq.(13) and with corresponding integral representations given by

$$
\begin{aligned}
V^{H}_{12i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{H}\, y\,(1-x)\,(1-y)\, R_{12i;H}\, \chi_{H}^{-2-\epsilon},\ i \neq 4\,, \\
V^{H}_{124} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{H}\,(1-x)\,\chi_{H}^{-1-\epsilon}\,, \qquad R_{12i;H} = -\,R_{22i;H}. \qquad (266)
\end{aligned}
$$

Contracting both sides of Eq.(265) with $p_{1\nu}$ and $p_{2\mu}$ we obtain the following set of four equations:

$$
\begin{aligned}
p_1^2\, V^{H}_{121} + p_{12}\ V^{H}_{123} + V^{H}_{124} &= \frac{1}{2}\Big[l_{156}\, V^{H}_{11}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) \\
&-\ V^{G}_{12}(p_2, p_2, -p_1, \{m\}_{21634}) -\ V^{G}_{12}(-p_2, -p_2, p_1, \{m\}_{12543})\Big], \\
p_1^2\, V^{H}_{123} + p_{12}\, V^{H}_{122} &= \frac{1}{2}\Big[l_{156}\, V^{H}_{12}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) +\ V^{G}_{11}(p_2, p_2, -p_1, \{m\}_{21634}) \\
&+\ V^{G}_{11}(-p_2, -p_2, p_1, \{m\}_{12543}) -\ V^{G}_{12}(p_2, p_2, -p_1, \{m\}_{21634}) \\
&-\ V^{G}_{12}(-p_2, -p_2, p_1, \{m\}_{12543}) +\ V^{G}_{0}(p_2, p_2, -p_1, \{m\}_{21634})\Big], \\
p_2^2\, V^{H}_{123} + p_{12}\, V^{H}_{121} &= \frac{1}{2}\Big[l_{212}\, V^{H}_{21}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) +\ V^{G}_{11}(p_1, p_1, -p_2, \{m\}_{56234}) \\
&+\ V^{G}_{0}(-P, -P, -p_2, \{m\}_{34256}) -\ V^{G}_{12}(p_1, p_1, -p_2, \{m\}_{56234})\Big], \\
p_2^2\, V^{H}_{122} + p_{12}\, V^{H}_{123} + V^{H}_{124} &= \frac{1}{2}\Big[l_{212}\, V^{H}_{22}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}) +\ V^{G}_{21}(-P, -P, -p_2, \{m\}_{34256}) \\
&+\ V^{G}_{0}(-P, -P, -p_2, \{m\}_{34256}) -\ V^{G}_{12}(p_1, p_1, -p_2, \{m\}_{56234})\Big].
\end{aligned}
$$

(267)

Note that the form factor V^{H}_{124} could be expressed in terms of generalized scalar functions; indeed we have

$$V^{H}_{124} = \frac{\omega^2}{2}\Big[V_{H}^{1,1|1,1|1,2}(n = 6 - \epsilon) + V_{H}^{1,1|1,1|2,1}(n = 6 - \epsilon)\Big], \tag{268}$$

while the remaining ones can be obtained solving the corresponding system of equations. Results for this family are summarized in Appendix B.6. $V^{H}_{0} \equiv V^{222}_{0}$ is discussed in Sect. 10.4 of III (see comment at the end of Section 9.1.2), evaluation of form factors in Section 11.4.

9.7 Integral representation for tensor integrals of rank three

Our aim in this work was to derive all the ingredients needed for the two-loop renormalization of the standard model (or of any other renormalizable theory) and to discuss all the tensor integrals that are relevant for the calculation of physical observables related to processes of the type $V(S) \to \overline{f}f$. For the classes of diagrams involving at least one four-point vertex, it is sufficient to analyze tensor integral that include up to two integration momenta in the numerator. However, for the remaining classes, V^{M}, V^{K}, and V^{H} it is necessary to consider in addition tensor integrals that include up to three momenta. As specified in the Introduction, we make use of the following shorthand notation: $\overline{x} = 1 - x, \overline{x}_i = 1 - x_i, \overline{y}_i = 1 - y_i$, etc.

9.7.1 V^{M} family

For general definitions see Section 9.3. We start by considering the integral with three uncontracted q_2 momenta in the numerator:

$$\begin{aligned} V^{M}(0|\alpha,\beta,\gamma;\cdots) = {}& V^{M}_{2221}\,\{\delta\, p_1\}_{\alpha\beta\gamma} + V^{M}_{2222}\,\{\delta\, p_2\}_{\alpha\beta\gamma} + V^{M}_{2223}\,\{p_1 p_1 p_2\}_{\alpha\beta\gamma} + V^{M}_{2224}\,\{p_2 p_2 p_1\}_{\alpha\beta\gamma} \\ & + V^{M}_{2225}\, p_{1\alpha}\, p_{1\beta}\, p_{1\gamma} + V^{M}_{2226}\, p_{2\alpha}\, p_{2\beta}\, p_{2\gamma}\,, \end{aligned} \tag{269}$$

where we used the definitions of Eq.(13). The various form factors have the following integrals representations (with integration measure defined in Eq.(177)):

$$\begin{aligned} V^{M}_{222i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{M}\, P_{222i;M}\,\chi_{M}^{-2-\epsilon}, \quad i > 2, \\ P_{2223;M} &= -z_1^2\, z_2, \quad P_{2224;M} = -z_1\, z_2^2, \quad P_{2225;M} = -z_1^3, \quad P_{2226;M} = -z_2^3, \\ V^{M}_{222i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{M}\, P_{222i;M}\,\chi_{M}^{-1-\epsilon}, \quad i = 1,2, \qquad P_{222i;M} = -z_i\,. \end{aligned} \tag{270}$$

$\chi_{M} \equiv \chi_{I}$ is given in Eq.(157). For the tensor integral with three uncontracted q_1 momenta in the numerator we use a decomposition identical to the one of Eq.(269). The integral representation for the corresponding form factors is given by

$$\begin{aligned} V^{M}_{111i} &= -\Gamma\,(2+\epsilon)\int \mathcal{D}V_{M} P_{111i;M}\,\chi_{M}^{-2-\epsilon}, \quad P_{111i;M} = x^3\, P_{222i;M}\,, \qquad i > 2, \\ V^{M}_{222i} &= -\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{M} x^2\left[P_{111i;M} + \frac{R_{111i;M}}{2-\epsilon} + 2\,\frac{1+\epsilon}{2-\epsilon}\,\chi_{M}^{-1}\, Q_{111i;M}\right]\chi_{M}^{-1-\epsilon}, \quad i = 1,2, \\ Q_{111i;M} &= \overline{x} z_i[F(z_1,z_2) + m_x^2], \qquad R_{111i;M} = (6-\epsilon)\,\overline{x} z_i, \quad P_{111i;M} = -x\, z_i. \end{aligned} \tag{271}$$

Employing again definitions analogous to those of Eq.(269), the form factors for the V^{M}_{122i} family are written as

$$\begin{aligned} V^{M}_{122i} &= -\Gamma\,(2+\epsilon)\int \mathcal{D}V_{M} P_{122i;M}\,\chi_{M}^{-2-\epsilon}, \qquad P_{122i;M} = x\, P_{222i;M}\,, \quad i > 2, \\ V^{M}_{122i} &= -\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{M} P_{122i;M}\,\chi_{M}^{-1-\epsilon}, \qquad P_{122i;M} = x\, P_{222i;M} \quad i = 1,2\,. \end{aligned} \tag{272}$$

Since the tensor integral $V^M(\alpha,\beta|\gamma)$ is only symmetric with respect to the exchange of the first two indices, a new decomposition in form factors is introduced:

$$
\begin{aligned}
V^M(\alpha,\beta|\gamma;\cdots) &= V^M_{1121}\,\{\delta\,p_1\}_{\alpha\beta\gamma} + V^M_{1122}\,\{\delta\,p_2\}_{\alpha\beta\gamma} + V^M_{1123}\,\{p_1p_1p_2\}_{\alpha\beta\gamma} + V^M_{1124}\,\{p_2p_2p_1\}_{\alpha\beta\gamma} \\
&+ V^M_{1125}\,p_{1\alpha}\,p_{1\beta}\,p_{1\gamma} + V^M_{1126}\,p_{2\alpha}\,p_{2\beta}\,p_{2\gamma} + V^M_{1127}\,\{\delta\,p_1\}_{\alpha\beta\,|\,\gamma} + V^M_{1128}\,\{\delta\,p_2\}_{\alpha\beta\,|\,\gamma}\,, \qquad (273)
\end{aligned}
$$

where all the symmetrized products were defined in Eq.(13). The integral representation for the form factors in Eq.(273) is as follows:

$$
\begin{aligned}
V^M_{112i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_M\,P_{112i;M}\,\chi_M^{-2-\epsilon}, \qquad P_{112i;M} = x^2\,P_{222i;M}, \quad i \neq 1,2,7,8, \\
V^M_{112i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_M\,x\,\Big[P_{112i;M} + \frac{R_{112i;M}}{2-\epsilon} + 2\,\frac{1+\epsilon}{2-\epsilon}\,\chi_M^{-1}\,Q_{112i;M}\Big]\,\chi_M^{-1-\epsilon} \quad i = 1,2, \\
Q_{112i;M} &= \overline{x}\,z_i\,[F(z_1,z_2)+m_x^2], \qquad R_{112i;M} = (6-\epsilon)\,\overline{x}\,z_i, \quad P_{112i;M} = -x\,z_i\,, \\
V^M_{112i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2-\epsilon}\int \mathcal{D}V_M\,x\,\overline{x}\,\chi_M^{-1-\epsilon}\,\Big[\frac{1}{2}\,R_{112i;M} + (1+\epsilon)\,\chi_M^{-1}\,Q_{112i;M}\Big], \quad i = 7,8, \\
Q_{1127;M} &= -\,z_1\,[F(z_1,z_2)+m_x^2], \qquad Q_{1128;M} = -\,z_2\,[F(z_1,z_2)+m_x^2], \\
R_{1127;M} &= -(6-\epsilon)\,z_1, \quad R_{1128;M} = -(6-\epsilon)\,z_2. \qquad (274)
\end{aligned}
$$

It is straightforward to show that the form factors V^M_{1113}, V^M_{1114}, V^M_{1115}, and V^M_{1116}, are generalized integrals of the type $V_M^{\alpha_1|\alpha_2,\alpha_3,\alpha_4|\alpha_5}$:

$$
\begin{aligned}
V^M_{1115} &= -36\,\omega^6\,V_M^{1|2,4,1|4}(n=10-\epsilon) - V^M_{1116} + 3\,V^M_{1114} - 3\,V^M_{1113}\,, \\
V^M_{1113} &= -12\,\omega^6\,V_M^{1|2,3,2|4}(n=10-\epsilon) - V^M_{1116} - 2\,V^M_{1114}\,, \\
V^M_{1114} &= -12\,\omega^6\,V_M^{1|2,2,3|4}(n=10-\epsilon) + V^M_{1116}\,, \quad V^M_{1116} = -36\,\omega^6\,V_M^{1|2,1,4|4}(n=10-\epsilon)\,. \qquad (275)
\end{aligned}
$$

9.7.2 V^K family

For general definitions see Section 9.5. We start by considering the tensor integrals $V(\mu,\nu,\alpha|0;\cdots)$ and $V(0|\mu,\nu,\alpha;\cdots)$ which are obviously totally symmetric and for which we can then adopt the same decomposition in form factors already presented in Eq.(269). Employing the standard procedure, one finds that

$$
\begin{aligned}
V^K_{222i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_K\,R_{222i;K}\,\chi_K^{-2-\epsilon}, \qquad i > 2\,, \\
R_{2223;K} &= Y_1\,Y_2^2, \quad R_{2224;K} = Y_1^2\,Y_2\,, \quad R_{2225;K} = Y_2^3\,, \quad R_{2226;K} = Y_1^3\,, \\
V^K_{222i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_K\,R_{222i;K}\,\chi_K^{-1-\epsilon}\,, \quad i = 1,2\,, \\
R_{2221;K} &= Y_2\,, \quad R_{2222;K} = Y_1\,, \qquad (276)
\end{aligned}
$$

where we recall that the quantities Y_1, Y_2 (see Eq.(11)) are given by $Y_i = -1 + y_i - y_3\,X$, with $X = (1-x_1)/(1-x_2)$. The integration measure is defined in Eq.(224). For the V^K_{111i} form factors we have

$$
\begin{aligned}
V^K_{111i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_K\,R_{111i;K}\,\chi_K^{-2-\epsilon}, \qquad i > 2\,, \\
R_{1113;K} &= -H_1\,H_2^2, \quad R_{1114;K} = -H_1^2\,H_2, \quad R_{1115;K} = -H_2^3\,, \quad R_{1116;K} = -H_1^3\,, \\
V^K_{111i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_K\,x_2\,\Big(x_2\,R_{111i;K} - Q_{111i;K}\Big)\,\chi_K^{-1-\epsilon}\,, \quad i = 1,2\,, \\
R_{1111;K} &= -\,H_2, \quad R_{1112;K} = -\,H_1, \quad Q_{1111;K} = \frac{\overline{x}_2}{y_3}H_2, \quad Q_{1112;K} = \frac{\overline{x}_2}{y_3}H_1. \qquad (277)
\end{aligned}
$$

The quantities H_1 and H_2 were introduced in Eq.(11). Consider now the integral $V^{K}(\mu|\nu,\alpha)$; in this case we have symmetry in the last two indices and a larger number of form factors; with symmetrized products defined in Eq.(13) we have

$$\begin{aligned}
V^{K}(\mu|\nu,\alpha;\cdots) &= V^{K}_{1221}\,\{\delta\,p_1\}_{\nu\alpha\,|\,\mu} + V^{K}_{1222}\,\{\delta\,p_2\}_{\nu\alpha\,|\,\mu} + V^{K}_{1223}\,\delta_{\nu\alpha}p_{1\mu} + V^{K}_{1224}\,\delta_{\nu\alpha}p_{2\mu} \\
&+ V^{K}_{1225}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu} + V^{K}_{1226}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^{K}_{1227}\,\{p_1\,p_1\,p_2\}_{\alpha\nu\,|\mu} \\
&+ V^{K}_{1228}\,\{p_2\,p_2\,p_1\}_{\alpha\nu\,|\,\mu} + V^{K}_{1229}\,p_{1\alpha}\,p_{1\nu}\,p_{2\mu} + V^{K}_{12210}\,p_{1\mu}\,p_{2\nu}\,p_{2\alpha}\,. \qquad (278)
\end{aligned}$$

The integral representations of the form factors of Eq.(278) are given by

$$\begin{aligned}
V^{K}_{122i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{K}\,R_{122i;K}\,\chi_{K}^{-2-\epsilon}, \qquad i>4\,, \\
R_{1225;K} &= -Y_2^2\,H_2, \quad R_{1226;K} = -Y_1^2\,H_1, \quad R_{1227;K} = -Y_1\,Y_2\,H_2, \\
R_{1228;K} &= -Y_1\,Y_2\,H_1, \quad R_{1229;K} = -Y_2^2\,H_1, \quad R_{12210;K} = -Y_1^2\,H_2, \\
V^{K}_{122i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{K}\,R_{122i;K}\,\chi_{K}^{-1-\epsilon}, \quad i=1,\cdots 4\,, \\
R_{1221;K} &= 1-x_1-H_2, \quad R_{1222;K} = 1-x_1-H_1, \quad R_{1223;K} = -H_2, \quad R_{1224;K} = -H_1. \qquad (279)
\end{aligned}$$

The integral $V^{K}(\mu,\nu|\alpha;\cdots)$ is symmetric in the first two indexes; using the definitions of Eq.(13) we obtain

$$\begin{aligned}
V^{K}(\mu,\nu|\alpha;\cdots) &= V^{K}_{1121}\{\delta\,p_1\}_{\mu\nu\,|\,\alpha} + V^{K}_{1122}\,\{\delta\,p_2\}_{\mu\nu\,|\,\alpha} + V^{K}_{1123}\,\delta_{\mu\nu}p_{1\alpha} + V^{K}_{1124}\,\delta_{\mu\nu}p_{2\alpha} \\
&+ V^{K}_{1125}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu} + V^{K}_{1126}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^{K}_{1127}\,\{p_1p_1p_2\}_{\mu\nu\,|\,\alpha} \\
&+ V^{K}_{1128}\,\{p_2p_2p_1\}_{\nu\mu\,|\,\alpha} + V^{K}_{1129}\,p_{1\mu}\,p_{1\nu}\,p_{2\alpha} + V^{K}_{11210}\,p_{1\alpha}\,p_{2\mu}\,p_{2\nu}\,. \qquad (280)
\end{aligned}$$

The integral representation of the form factors in Eq.(280) is the following:

$$\begin{aligned}
V^{K}_{112i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{K}\,R_{112i;K}\,\chi_{K}^{-2-\epsilon}, \qquad i>4\,, \\
R_{1125;K} &= Y_2\,H_2^2\,, \quad R_{1126;K} = Y_1\,H_1^2\,, \quad R_{1127;K} = Y_2\,H_1\,H_2\,, \\
R_{1128;K} &= Y_1\,H_1\,H_2\,, \quad R_{1129;K} = Y_1H_2^2\,, \quad R_{11210;K} = Y_2\,H_1^2, \\
V^{K}_{112i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{K}\,x_2\,R_{112i;K}\,\chi_{K}^{-1-\epsilon}, \quad i=1,2\,, \\
R_{1121;K} &= -\,H_2\,, \quad R_{1122;K} = -\,H_1\,, \\
V^{K}_{112i} &= -\,\frac{\Gamma\,(1+\epsilon)}{2}\int \mathcal{D}V_{K}\,x_2\,\Big[R_{112i;K} + y_3^{-1}\,\overline{x}_2\,Q_{112i;K}\Big]\,\chi_{K}^{-1-\epsilon}, \quad i=3,4\,, \\
R_{1123;K} &= 1-x_1-H_2\,, \quad R_{1124;K} = 1-x_1-H_1\,, \quad Q_{1123;K} = Y_2\,, \quad Q_{1124;K} = Y_1\,. \qquad (281)
\end{aligned}$$

9.7.3 V^{H} family

For general definitions see Section 9.6. We finally analyze the rank three tensor integrals in the family V^{H}. The tensor integrals $V^{H}(\mu,\nu,\alpha|0)$ and $V^{H}(0|\mu,\nu,\alpha)$ can be decomposed into form factors in complete analogy with Eq.(269). We provide here the integral representations for these form factors,

$$\begin{aligned}
V^{H}_{222i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{H}\,y^3\,R_{222i;H}\,\chi_{H}^{-2-\epsilon}, \qquad i>2\,, \\
R_{2223;H} &= -\,(z_2-z_3)^2\,(1-z_1-z_2)\,, \quad R_{1114;H} = (z_2-z_3)\,(1-z_1-z_2)^2\,, \\
R_{2225;H} &= (z_2-z_3)^3\,, \quad R_{2226;H} = -\,(1-z_1-z_2)^3\,, \\
V^{H}_{222i} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{H}\,y\,R_{222i;H}\,\chi_{H}^{-1-\epsilon}, \quad i=1,2\,, \\
R_{2221;H} &= z_2-z_3\,, \quad R_{2222;H} = -\,(1-z_1-z_2)\,, \qquad (282)
\end{aligned}$$

where the integration measure is given in Eq.(253). Also

$$
\begin{aligned}
V^{H}_{111i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{K}\,\overline{x}^{3}\,\overline{y}^{3}\,R_{111i;H}\,\chi_{H}^{-2-\epsilon}, \qquad R_{111i;H} = -\,R_{222i;H}, \quad i>2\,,\\
V^{H}_{111i} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{H}\,\overline{x}^{2}\,\overline{y}\,\Big[\overline{x}\,R_{111i;H} + \frac{x}{y}\,Q_{111i;H}\Big]\,\chi_{H}^{-1-\epsilon}\,, \quad i=1,2,\\
R_{1111;H} &= Q_{1111;H} = z_3 - z_2\,, \quad R_{1112;H} = Q_{1112;H} = 1 - z_1 - z_2\,.
\end{aligned}
\tag{283}
$$

For the tensor integral $V^{H}(\mu|\nu,\alpha)$ we employ another decomposition into form factors, based on the definitions of Eq.(13):

$$
\begin{aligned}
V^{H}(\mu|\nu,\alpha;\cdots) = {}& V^{H}_{1221}\{\delta\,p_1\}_{\nu\alpha\,|\,\mu} + V^{H}_{1222}\,\{\delta\,p_2\}_{\nu\alpha\,|\,\mu} + V^{H}_{1223}\,\delta_{\nu\alpha}p_{1\mu} + V^{H}_{1224}\,\delta_{\nu\alpha}p_{2\mu} + V^{H}_{1225}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu}\\
&+ V^{H}_{1226}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^{H}_{1227}\,\{p_1\,p_1\,p_2\}_{\mu\nu\alpha} + V^{H}_{1228}\,\{p_1\,p_2\,p_2\}_{\mu\nu\alpha}\,,
\end{aligned}
\tag{284}
$$

obtaining the following parametrization:

$$
\begin{aligned}
V^{H}_{122i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{H}\,y^{2}\,\overline{x}\,\overline{y}\,R_{122i;H}\,\chi_{H}^{-2-\epsilon}, \qquad i>4\,,\\
R_{1225;H} &= -\,(z_2-z_3)^3, \quad R_{1226;H} = (1-z_1-z_2)^3,\\
R_{1227;H} &= (z_2-z_3)^2\,(1-z_1-z_2), \quad R_{1228;H} = -\,(z_2-z_3)\,(1-z_1-z_2)^2,\\
V^{H}_{122i} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{H}\,\overline{x}\,R_{122i;H}\,\chi_{H}^{-1-\epsilon}\,, \quad i\le 4\,,\\
R_{1221;H} &= y\,(z_2-z_3), \quad R_{1222;H} = -\,y\,(1-z_1-z_2),\\
R_{1223;H} &= -\,\overline{y}\,(z_2-z_3), \quad R_{1224;H} = \overline{y}\,(1-z_1-z_2)\,.
\end{aligned}
\tag{285}
$$

Finally, for the tensor integral $V^{H}(\mu,\nu|\alpha)$ we adopt the decomposition

$$
\begin{aligned}
V^{H}(\mu,\nu|\alpha;\cdots) = {}& V^{H}_{1121}\{\delta\,p_1\}_{\mu\nu\,|\,\alpha} + V^{H}_{1122}\,\{\delta\,p_2\}_{\mu\nu\,|\,\alpha} + V^{H}_{1123}\,\delta_{\mu\nu}p_{1\alpha} + V^{H}_{1124}\,\delta_{\mu\nu}p_{2\alpha} + V^{H}_{1125}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu}\\
&+ V^{H}_{1126}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^{H}_{1127}\,\{p_1\,p_1\,p_2\}_{\mu\nu\alpha} + V^{H}_{1128}\,\{p_1\,p_2\,p_2\}_{\mu\nu\alpha}\,,
\end{aligned}
\tag{286}
$$

where symmetrized products are defined in Eq.(13). We obtain the corresponding expression for the form factors:

$$
\begin{aligned}
V^{H}_{112i} &= -\,\Gamma\,(2+\epsilon)\int \mathcal{D}V_{H}\,y\,\overline{x}^{2}\,\overline{y}^{2}\,R_{112i;H}\,\chi_{H}^{-2-\epsilon}, \qquad R_{112i;H} = -\,R_{122i;H}, \quad i>4\,,\\
V^{H}_{112i} &= -\,\frac{1}{2}\,\Gamma\,(1+\epsilon)\int \mathcal{D}V_{H}\,\overline{x}\,R_{112i;H}\,\chi_{H}^{-1-\epsilon}\,, \quad i=1\cdots 4\,,\\
R_{1121;H} &= -\overline{x}\,\overline{y}\,(z_2-z_3)\,, \quad R_{1122;H} = \overline{x}\,\overline{y}\,(1-z_1-z_2)\,,\\
R_{1123;H} &= (1-\overline{x}\,\overline{y})\,(z_2-z_3)\,, \quad R_{1124;H} = -\,(1-\overline{x}\,\overline{y})\,(1-z_1-z_2)\,.
\end{aligned}
\tag{287}
$$

Note that $V^{H}(\mu|\nu,\alpha)$ and also $V^{H}(\mu,\nu|\alpha)$ require a smaller number of form factors than $V^{K}(\mu|\nu,\alpha)$ and $V^{K}(\mu,\nu|\alpha)$. One can check that this is indeed the case by repeating the arguments already used in discussing $V^{H}(\mu|\nu)$.

We conclude observing that another way of parametrizing rank three tensors is through Eq.(48), after which the corresponding form factors are obtained with the help of Eq.(51); the two sets of form factors are easily related but with this parametrization and for a singular Gram matrix the inversion can be done with its pseudo-inverse, as pointed out in [22].

9.8 Diagrammatic interpretation of the reduction procedure

All the manipulations discussed in the previous Sections, aimed at reducing form factors to combinations of scalar integrals, have a diagrammatic counterpart. Diagrams with reducible scalar products in the numerator give rise to standard scalar functions of the same family and contractions corresponding to diagrams

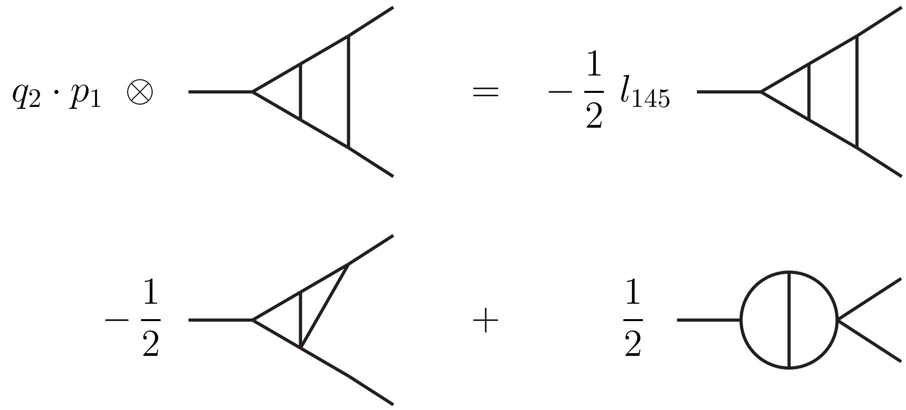


Figure 12: Diagrammatic interpretation of the reduction induced by a reducible scalar product. Here $l_{145} = p_1^2 - m_4^2 + m_5^2$, while the symbol $\otimes$ denotes insertion of a scalar product into the numerator of the diagram.

with fewer internal lines, as illustrated in Fig. 12 (there, the symbol $\otimes$ denotes insertion of a scalar product into the numerator of the diagram). The figure is based on the simple relation $2\, q_2 \cdot p_1 = [5]_K - [4]_K - l_{145}$. After permutation of momenta we obtain the first of Eqs. (228) where the form factors are expressed in their standard form.

There are $7 - I$ irreducible scalar products for two-loop vertices, neglecting additional branching of the external lines (as in Fig. 5), I being the number of internal lines in the graph; in the reduction procedure they give raise to both contractions, i.e. scalar diagrams with less propagators, and to ordinary/generalized scalar functions of the same family as illustrated in Fig. 13. The component with contractions and ordinary

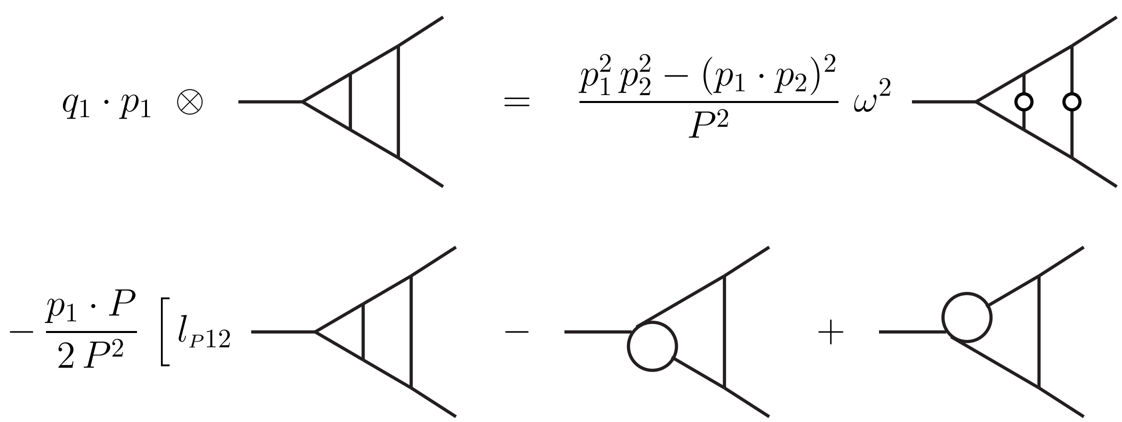
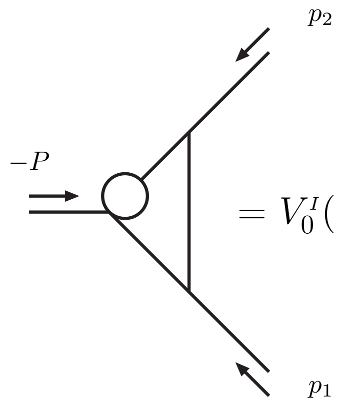


Figure 13: Diagrammatic interpretation of the reduction induced by an irreducible scalar product. In the first diagram of the RHS non-canonical powers -2 in propagators are explicitly indicated by a circle and the space-time dimension is $6 - \epsilon$. Here $l_{P12} = P^2 - m_1^2 + m_2^2$ and $\omega = \mu^2/\pi$ where μ is the unit of mass. The symbol $\otimes$ denotes insertion of a scalar product into the numerator of the diagram.

scalar functions is given in the second row of Fig. 13 while the irreducible component is expressed through a generalized scalar function in $6 - \epsilon$ space-time dimension, as depicted in the first row of Fig. 13 (there, a circle denotes a non-canonical power 2 for the corresponding propagator). Note that the irreducible component appears multiplied by the Gram determinant.

Whenever this relation, or similar ones, is used in the reduction procedure, the last diagram on the r.h.s. of Fig. 13 will be written as in Eq.(228) after a rearrangement of its arguments, see Fig. 14. In principle a



$$= V_0^I(-p_2\ - P\,,\ \{m\}_{23654})$$

Figure 14: Rearrangement of arguments bringing the diagram in the l.h.s. to the standard form of the I-family, see Section 9.2.

generalized scalar function can be cast into the form of a combination of ordinary scalar functions using IBPI techniques but, in practice, these solutions are poorly known in the fully massive case; it is somehow hard to accept that part of our present limitations are related to a poor level of technical handling of large systems of linear equations; however, this really represents the bottleneck of many famous approaches (see [32] for recent developments).

10 Graphs, form factors and permutations

Diagrams of any renormalizable field theory, like the standard model, must be generated according to the rules of the theory itself, they must be assembled to construct some physical amplitude and a reduction must be performed. There are many technical details hidden in this procedure, in particular some efficient way of handling the different topologies while assembling the grand total of diagrams.

We briefly illustrate our approach: for the sake of clarity we refer to the V^E-family. In principle, for a fixed choice of the external momenta we should consider three kind of diagrams, as shown in Fig. 15. However,

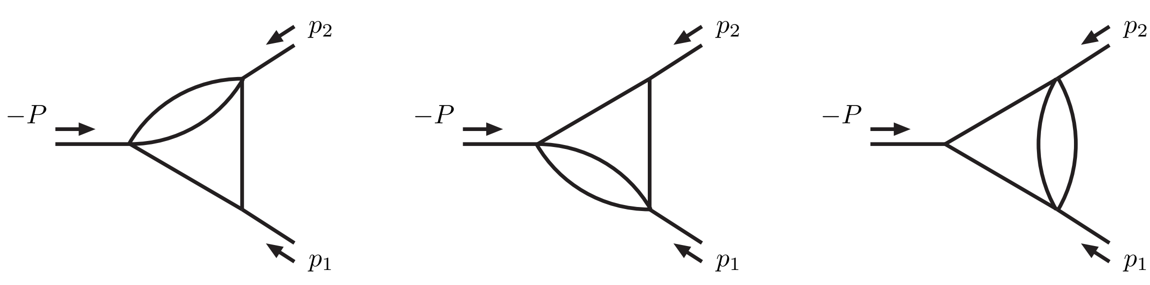


Figure 15: The V^E-family. External momenta flow inwards.

in our automatized procedure, we will only compute the first diagram of Fig. 15 since the remaining two are obtainable through permutation of the external momenta. To illustrate the procedure we consider a specific example, the process $H(-P) + \gamma(p_1) + \gamma(p_2) \to 0$; in the standard model there will be diagrams like the one of Fig. 16(a) which can be expressed as a combinations of functions $V^{E}_{i\cdots j}(p_2, P, M_{_W}, M_{_H}, M_{_W}, M_{_W})$, but also diagrams like in Fig. 16(b) which are always evaluated according to the conventions of Fig. 16(c). Therefore

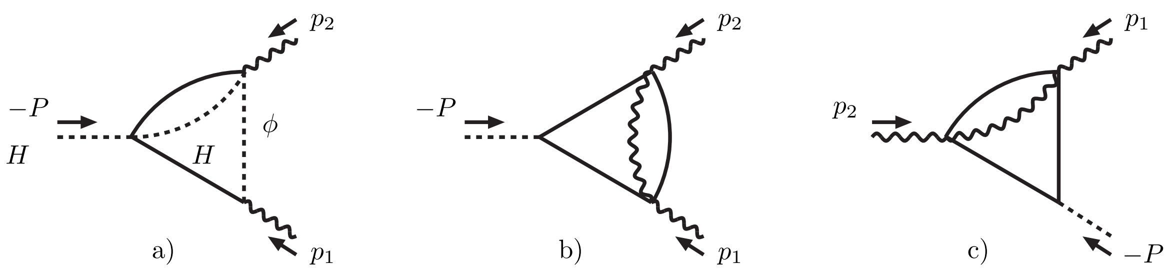


Figure 16: A V^{E}-family contribution to $H(-P) + \gamma(p_1) + \gamma(p_2) \to 0$. External momenta flow inwards.

they correspond to combinations of functions

$$V^{E}_{i\cdots j}(p_1, -p_2, M_{_W}, 0, M_{_W}, M_{_W}). \tag{288}$$

Similarly the decomposition into form factors will be as follows:

$$V^{E}(\mu\,|\,0\,;\,p_1, -p_2, M_{_W}, 0, M_{_W}, M_{_W}) = -\,V^{E}_{11}(p_1, -p_2, \cdots)\,P_\mu + V^{E}_{12}(p_1, -p_2, \cdots)\,p_{1\mu}, \tag{289}$$

etc, showing that the consistent basis to expand the form factors is $(-P\,,\,p_1)$. After permutation, the results of our paper follow automatically. For a correct treatment of the combinatorial factors we refer the reader to Appendix C.

11 Strategies for the evaluation of two-loop vertices

Scalar configurations for irreducible two-loop vertices were considered and evaluated in III, where tables of numerical results were presented. The techniques include several variations of the standard BT-algorithm [17] and the introduction of parameter-dependent C-functions (for which we refer the reader to Appendix E of III, where they are introduced and their numerical evaluation is discussed in Eqs. (291-294)).

The same set of procedures can be easily generalized to cover a non-trivial theory (i.e. one with spin) using the defining parametric representations and the reduction formalism derived in this article.

A few relevant examples will be shown and discussed in the following sections. We will place special emphasis on proving that new ultraviolet poles, not present in scalar configurations, do not prevent the derivation of representations of the class Eq.(2) for tensor integrals.

The three diagrams (with non-trivial numerators) belonging to the V^{1N1}-family, namely V^{E}, V^{I} and V^{M}, are evaluated by repeated applications of the BT algorithm [17]; in this case the procedure remains the same as for scalar configurations, since the BT-algorithm works independently of the presence of additional polynomials of Feynman parameters in the numerator. We only have to pay some attention to the limit $\epsilon \to 0$, which cannot be taken from the very beginning for tensor configurations that are ultraviolet divergent. The introduction of parameter-dependent C-functions for tensor integrals of the remaining families is also shown.

11.1 Examples in the V^{E}-family

A typical example is given by the V^{E}-family where we can easily provide integral representations for the scalar representative and for the form factors, e.g.

$$V^{E}_{0} = -\frac{1}{\epsilon^2} - \overline{\Delta}^2_{UV} + \overline{\Delta}_{UV}\left[2\int dC_2\,\ln\chi_{_E}(x, 1, y) - 1\right]$$

$$
+ \int dCS\,(x\,;\,y,z)\; \frac{\ln\chi_{E}(x,y,z)}{1-y}\Big|_{+} + \int dC_2\; \ln\chi_{E}(x,1,y)\,L_{E}(x,y) - \frac{3}{2} - \frac{1}{2}\,\zeta(2),
$$

$$
V^{E}_{a} = -\,\frac{1}{2}\left[\frac{1}{\epsilon^2} - \overline{\Delta}^{2}_{UV}\right] - \frac{3}{16} - \frac{1}{4}\,\zeta(2) + 2\,\overline{\Delta}_{UV}\left[\int dC_2\,(1-y)\;\ln\chi_{E}(x,1,y) + \frac{1}{8}\right]
$$

$$
- \int dCS\,(x\,;\,y,z)\;\Big[\ln\chi_{E}(x,y,z) + (1-z)\,\frac{\ln\chi_{E}(x,y,z)}{1-y}\Big|_{+}
$$

$$
+ \int dC_2\,(1-y)\;\ln\chi_{E}(x,1,y)\,L_{E}(x,y),
$$

$$
V^{E}_{b} = \frac{1}{\epsilon^2} + \overline{\Delta}^{2}_{UV} - 2\,\overline{\Delta}_{UV}\int dC_2\;\ln\chi_{E}(x,1,y) + \int dCS\,(x\,;\,y,z)\;\ln\chi_{E}(x,y,z)
$$

$$
- \int dCS\,(x\,;\,y,z)\;\frac{\ln\chi_{E}(x,y,z)}{1-y}\Big|_{+} - \int dC_2\;\ln\chi_{E}(x,1,y)\,L_{E}(x,y) + 1 + \frac{1}{2}\,\zeta(2), \tag{290}
$$

$$
V^{E}_{a} = V^{E}_{21} - V^{E}_{22}, \qquad V^{E}_{b} = V^{E}_{22}, \tag{291}
$$

where the l.h.s. of the last equation refers to the (p_1, P) basis. Furthermore, $L_{E}(x,y) = \ln(1-y) - \ln x - \ln(1-x) - \ln\chi_{E}(x,1,y)$ and χ_{E} is obtained from Eq.(125) by rescaling by $1/\mid P^2\mid$.

Smooth integral representations for higher tensors can be classified according to

$$
V^{E}_{i} = K_i + a_i\,\overline{\Delta}_{UV}\int dC_2\,y^{\alpha_i}\;\ln\chi_{E}(x,1,y) + b_i\int dCS\,(x\,;\,y,z)\;z^{\beta_i}\,\frac{\ln\chi_{E}(x,y,z)}{1-y}\Big|_{+}
$$

$$
+ \int dCS\,(x\,;\,y,z)\;P_i(y,z)\;\ln\chi_{E}(x,y,z) + c_i\int dC_2\,y^{\gamma_i}\;\ln\chi_{E}(x,1,y)\,L_{E}(x,y), \tag{292}
$$

and coefficients and exponents for the first few cases are reported in Tab. 1.

i	K	a	α	b	β	P	c	γ
221	$-\frac{2}{3}\,\overline{\Delta}_{UV}\,\epsilon^{-1} - \frac{1}{18}\,\overline{\Delta}_{UV} - \frac{1}{3}\,\Delta^2_{UV} - \frac{151}{216} - \frac{1}{6}\,\zeta(2)$	2	2	1	2	0	1	2
222	$-2\,\overline{\Delta}_{UV}\,\epsilon^{-1} + \frac{1}{2}\,\overline{\Delta}_{UV} - \Delta^2_{UV} - \frac{7}{8} - \frac{1}{2}\,\zeta(2)$	2	0	1	0	$-1-y$	1	0
223	$-\,\overline{\Delta}_{UV}\,\epsilon^{-1} - \frac{1}{2}\,\Delta^2_{UV} - \frac{3}{4} - \frac{1}{4}\,\zeta(2)$	2	1	1	1	$-z$	1	1

Table 1: Parameters for the V^{E} form factors according to Eq.(292).

11.2 Examples in the V^{I}, V^{M} families

Consider the V^{I}-family as a second example. All form factors can be expressed as linear combinations of integrals of the following kind (χ_{I} is obtained from Eq.(157) by rescaling by $1/\mid P^2\mid$):

$$
I_{0;I} = \int \mathcal{D}V_{I}\,\chi_{I}^{-1-\epsilon}, \quad I_{1x;I} = \int \mathcal{D}V_{I}\,\chi_{I}^{-1-\epsilon}\,x, \quad I_{2xx;I} = \int \mathcal{D}V_{I}\,\chi_{I}^{-1-\epsilon}\,x^2, \qquad \text{etc.} \tag{293}
$$

As an illustration we derive the ultraviolet finite part for the first few integrals of the list, introducing

$$I^{\rm fin}_{n;I} = \frac{1}{M^2\, b_I}\Big[\int_0^1 dxdy \int_0^{1-y} dz_1 \int_0^{z_1} dz_2\, I^4_{n;I} + \int_0^1 dxdy \int_0^{1-y} dz\, I^3_{n;I} + \int dC_2\, I^2_{n;I} + I^0_{n;I}\Big]. \quad (294)$$

Notation follows closely that of Sect. 6.1 of III (see also Eq. (12) of III for the definition of $[y,z,u]_i$) and b_I is the BT-factor of the function (see Eq. (64) of III); therefore we have

$$f(\{x\}\,;\,[y\,z\,u]_i) = \begin{cases} f(\{x\}\,;\,y,z) \text{ for } i=0 \\ f(\{x\}\,;\,z,z) \text{ for } i=1 \\ f(\{x\}\,;\,z,u) \text{ for } i=2 \end{cases}. \quad (295)$$

$$b_I = (\nu_1^2+\mu_3^2-\mu_4^2)^2 - \mu_{35}^2\,(1+\nu_1^2-\nu_2^2)\,(\nu_1^2+\mu_3^2-\mu_4^2) + \mu_{35}^4\,\nu_1^2 + \lambda_I\mu_3^2, \quad (296)$$

where we have set $\lambda_I = \lambda(1,\nu_1^2,\nu_2^2)$ and where

$$\mu_i^2 = \frac{m_i^2}{\mid P^2\mid}, \quad i=1,\dots,N, \qquad \nu_j^2 = \mid \frac{p_j^2}{P^2} \mid, \quad j=1,2, \qquad \mu_{ij}^2 = 1+\mu_i^2-\mu_j^2. \quad (297)$$

BT co-factors are

$$\begin{aligned} &Z_{0;I} = -\,\lambda_I, \qquad Z_{3;I}=0, \qquad Z_{1;I} = (1-\nu_1^2-\nu_2^2)\,(\nu_1^2-\mu_{45}^2) + 2\,(\nu_1^2+\mu_3^2-\mu_4^2)\,\nu_2^2 \\ &Z_{2;I} = -(1-\nu_1^2-\nu_2^2)\,(\nu_1^2+\mu_3^2-\mu_4^2) - 2\,(\nu_1^2-\mu_{45}^2)\,\nu_1^2, \quad Z^-_{i;I} = Z_{i;I} - Z_{i+1;I}. \end{aligned} \quad (298)$$

We define additional auxiliary functions:

$$\begin{aligned} &\xi_I(x,y,z_1,z_2) = \chi_I(x,1-y,z_1,z_2), \\ &L_I(x,y,z_1,z_2) = \ln(1-y) - \ln(x) - \ln(1-x) - \ln\xi_I(x,y,z_1,z_2). \end{aligned} \quad (299)$$

Our results are as follows:

$$\begin{aligned}
I^4_{0;I} &= -\frac{\ln\xi_I(x,y,z_1,z_2)}{y}\Big|_+, \\
I^3_{0;I} &= \Big[1-\,L_I(x,0,1-y,z)\Big]\,\ln\,\xi_I(x,0,1-y,z) + \frac{1}{2}\,\ln\xi_I(x,y,[1-y,z,0]_0) \\
&+\,\frac{1}{2}\,\sum_{i=0}^{2}\,Z^-_{i;I}\,\frac{\ln\xi_I(x,y,[1-y,z,0]_i)}{y}\Big|_+, \\
I^2_{0;I} &= \frac{1}{2}\,\sum_{i=0}^{2}\,Z^-_{i;I}\,\ln\xi_I(x,0,[1-y,z,0]_i)\,L_I(x,0,[1-y,z,0]_i), \qquad I^0_{0;I} = -\,\frac{1}{4}\,S_1(1) + \frac{1}{8}, \\
I^i_{1x;I} &= x\,I^i_{0;I}, \quad (i\neq 0), \qquad I^0_{1x;I} = -\,\frac{1}{8}\,S_1(2) + \frac{1}{8}, \qquad I^4_{1y;I} = -\,\ln\xi_I(x,y,z_1,z_2), \\
I^3_{1y;I} &= \frac{1}{2}\,\sum_{i=0}^{2}\,Z^-_{i;I}\,\ln\xi_I(x,y,[1-y,z,0]_i) + \frac{1}{2}\,y\,\ln\xi_I(x,y,[1-y,z,0]_0), \\
I^3_{1y;I} &= I^2_{1y;I} = 0, \qquad I^0_{1y;I} = -\,1, \qquad I^4_{1z_1;I} = \frac{1}{2}\,(Z_1 - 3\,z_1)\,\frac{\ln\xi_I(x,y,z_1,z_2)}{y}\Big|_+, \\
I^3_{1z_1;I} &= \frac{1}{2}\,(Z_1-3\,z)\,L_I(x,0,1-y,z)\,\ln\xi_I(x,0,1-y,z) + \frac{1}{2}\,(Z_1-y)\,\ln\xi_I(x,y,[1-y,z,0]_0) \\
&+\,z\,\ln\xi_I(x,0,1-y,z) + \frac{1}{2}\,Z^-_{0;I}\,\frac{\ln\xi_I(x,y,[1-y,z,0]_0)}{y}\Big|_+ + \frac{z}{2}\,Z^-_{1;I}\,\frac{\ln\xi_I(x,y,[1-y,z,0]_1)}{y}\Big|_+
\end{aligned}$$

$$
+ \frac{z}{2}\, Z^-_{2;I}\, \frac{\ln \xi_I(x,y,[1-y,z,0]_2)}{y}\Big|_+,
$$

$$
\begin{aligned}
I^2_{1z_1;I} &= \frac{1}{2}\, Z^-_{0;I}\, L_I(x,0,[1,1-y,0]_0)\, \ln \xi_I(x,0,[1,1-y,0]_0) \\
&+ \frac{1-y}{2} \sum_{i=1,2} Z^-_{i;I}\, L_I(x,0,[1,1-y,0]_i)\, \ln \xi_I(x,0,[1,1-y,0]_i),
\end{aligned}
$$

$$
I^0_{1z_1;I} = -\frac{1}{6}\, S_1(1) + \frac{5}{36}, \qquad I^4_{1z_2;I} = \frac{1}{2}\,(Z_2 - 3\, z_2)\, \frac{\ln \xi_I(x,y,z_1,z_2)}{y}\Big|_+,
$$

$$
\begin{aligned}
I^3_{1z_2;I} &= \frac{1}{2}\,(Z_2 - 3\, z)\, L_I(x,0,1-y,z)\, \ln \xi_I(x,0,1-y,z) + \frac{z_1}{2} \sum_{i=0}^{1} Z^-_{i;I}\, \frac{\ln \xi_I(x,y,[1-y,z,0]_i)}{y}\Big|_+ \\
&+ \frac{z}{2}\, \ln \xi_I(x,y,[1-y,z,0]_0) + z\, \ln \xi_I(x,0,1-y,z),
\end{aligned}
$$

$$
I^2_{1z_2;I} = \frac{1-y}{2} \sum_{i=0}^{1} Z^-_{i;I}\, L_I(x,0,[1,1-y,0]_i)\, \ln \xi_I(x,0,[1,1-y,0]_i), \quad I^0_{1z_2;I} = -\frac{1}{12}\, S_1(1) + \frac{5}{72}, \tag{300}
$$

where $S_n(k) = \sum_{l=1}^{k} l^{-n}$. Similar expressions can be written also for higher order form factors showing, once more, that scalar and tensor integrals give similar results and can be treated in one single stroke.

Once again the whole procedure can be described in terms of a specific example. Consider the diagram of Fig. 17 which contributes to the on-shell decay amplitude $Z \to l^+ l^-$. The on-shell vertex, including external wave-functions, is decomposed according to Eq.(35) and the corresponding coefficients are subsequently evaluated; for instance we consider the contribution coming from the diagrams of Fig. 17 and derive the vector coefficient in the limit $m_f = 0$. Using Eq.(38) and taking the trace we obtain the following expression

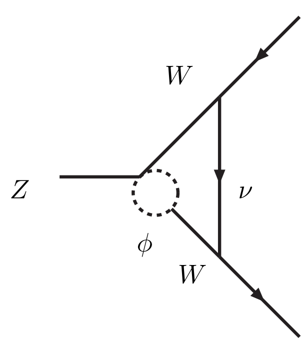


Figure 17: Diagram of the V^I-family contributing to $Z \to l^+ l^-$.

$$
\begin{aligned}
F_V = \frac{i\pi^4 g^5 s_\theta^2}{16 c_\theta} \Big[\, & M_Z^2\, V_0^I\,(p_1, P, M_Z, M_W, M_W, 0, M_W) \;+\; M_W^2\, V_{11}^I\,(p_1, P, M_Z, M_W, M_W, 0, M_W) \\
& + (\, M_W^2 \;+ M_Z^2\,)\, V_{12}^I\,(p_1, P, M_Z, M_W, M_W, 0, M_W) \;+\; A_0([M_W, M_Z])\, C_0(p_1, p_2, M_W, 0, M_W) \\
& -\; V_0^E\,(p_1, P, M_Z, M_W, 0, M_W) \;+\; V_0^E\,(0, P, M_Z, M_W, M_W, M_W) \\
& +\; 2\, V_{11}^E\,(0, P, M_Z, M_W, M_W, M_W) \;-\; V_{12}^E\,(p_1, P, M_Z, M_W, 0, M_W)\,\Big].
\end{aligned} \tag{301}
$$

The form factors of the V^E-family in Eq.(301) can be further reduced according to the results of Section 9.1 or, more conveniently, they can be computed according to Eqs.(118)–(123). A similar situation appears for the form factors of Eq.(301) of the V^I-family for which we use the reduction techniques of Section 9.2 or an explicit evaluation using Eqs.(293)–(300).

Results in the V^M-family are very similar in their structure and will not be reported here. Furthermore, the graph corresponds to a one-loop self-energy insertion which should be Dyson–re-summed.

11.3 Examples in the V^G-family

Coming back to the strategy to evaluate tensor integrals, we observe that two other scalar diagrams, V^G and V^K, were expressed in III in terms of integrals of C-functions (Appendix E of III).

It is very easy to extend the derivation to tensors. Consider, for instance, the V^G case: starting from Eq.(202) the appropriate strategy will be as follows. If we need to prove a WST identity, where the presence of Gram determinants is inessential, we simply invert the system and derive V^G_{ij} with $i,j = 1,2$ in terms of known quantities. If instead we need to use these form factors to compute some physical observable, then a possible strategy is the following: suppose that $p_1^2 \neq 0$, then V^G_{11} is eliminated and V^G_{12} is either given in terms of $V^{1,1|1,2|2}$ at $n = 6-\epsilon$ or explicitly evaluated.

If we choose the second strategy then χ_G is a quadratic form in y_1, y_2, with x-dependent coefficients and we can use the results of Appendix E of III to write

$$V^G_{12} = -\frac{1}{M^2}\int dS_2(\{x\})\, x_2 \Big[C_{11}(0) - C_0(0)\Big], \tag{302}$$

with $|P^2| = M^2$. Similarly, if $p_1 \cdot p_2 \neq 0$ we can eliminate V^G_{21} and express V^G_{22} in terms of generalized scalars as in Eq.(204), or explicitly derive

$$V^G_{22} = -\frac{1}{M^2}\int dS_2(\{x\})\Big[C_{11}(0) - C_0(0)\Big]. \tag{303}$$

For this family the rank two tensor integrals are ultraviolet divergent. For the form factors of the $22i$-family defined in Eq.(205) the relevant quantity is $V^G_{224} = V^{2,1|1,1|1}$ evaluated at $n = 6-\epsilon$ which, with χ_G obtained from Eq.(197) by rescaling by $1/\mid P^2 \mid$, can be rewritten according to

$$V^G_{224} = -\frac{1}{2}\Big(\frac{\omega}{M^2}\Big)^{\epsilon}\Gamma(\epsilon)\int dS_2(\{x\})\Big[x_2\,(1-x_2)\Big]^{-1-\epsilon}\int dS_2(\{y\})y_2^{\epsilon/2}\,\chi_G^{-\epsilon}. \tag{304}$$

Eq.(304) shows the expected ultraviolet poles; indeed the integral is overall ultraviolet divergent and so is the (β,γ) sub-diagram. With ω defined in Eq.(15) and $\chi_{G\,;\,0} = \chi_G(x_2 = 0)$ we obtain

$$\begin{aligned}
V^G_{224} &= \frac{1}{2}\int dS_2(x_1,x_2)\int dS_2(y_1,y_2)\,(1-x_2)^{-1}\,V^{G\,;\,4}_{224} + \frac{1}{2}\int dCS\,(x_1\,;\,y_1,y_2)\;V^{G\,;\,3}_{224} + K,\\
V^{G\,;\,4}_{224} &= \ln\chi_{G\,;\,0} + \frac{\ln\chi_G}{x_2}\Big|_+, \qquad V^{G\,;\,3}_{224} = -\ln\chi_{G\,;\,0}\Big(\overline{\Delta}_{UV} - \ln x_1 + \frac{1}{2}\ln y_2 - \frac{1}{2}\ln\chi_{G\,;\,0}\Big),\\
K &= \frac{1}{8}\Big(\frac{1}{\epsilon^2} + \overline{\Delta}^2_{UV}\Big) - \frac{3}{16}\overline{\Delta}_{UV} + \frac{7}{64} - \frac{1}{8}\zeta(2),
\end{aligned} \tag{305}$$

where, as usual, $\overline{\Delta}_{UV} = 1/\epsilon - \Delta_{UV}$ and $\Delta_{UV} = \gamma - \ln\omega/M^2$, with $M^2 =\mid P^2 \mid$. Similarly V^G_{114} will develop a double ultraviolet pole being overall divergent with (α,γ) divergent. In this case the additional pole is hidden in the y_2-integration, as shown in Eq.(217).

11.4 Examples in the V^K, V^H-families

Also the V^K-family can be expressed in terms of well-behaved integrals of C-functions, introduced in Appendix E of III. From Eq.(230) we see that one of the relevant objects to be evaluatet is $I_{R;K} = V^{1,1|1,2,1|2}$ for $n = 6-\epsilon$: the important result is that this quantity can be computed along the same lines of the corresponding scalar integral.

The derivation is straightforward: starting from Eq.(224) we will adopt the same technique as in Sect. 9.1 of III; with $X = (1-x_1)/(1-x_2) = 1 - \overline{X}$ we change variables according to $y_1 = y'_1 + X\,y_3, y_2 = y'_2 + X\,y_3$ and $y_3 = y'_3$. Next we perform the y_3 integration analytically; after that the $y_1 - y_2$ interval is mapped into the standard triangle $0 \leq y_2 \leq y_1 \leq 1$ and the net result is a combination of 10 integrals of C functions with $\{x\}$ dependent parameters, as defined in Tab. 2 of III. Therefore, we obtain expressions for both standard and generalized scalar as

$$
\begin{aligned}
V_{0}^{K} &= -\frac{1}{M^4}\int dS_2(x_1,x_2)\,\frac{x_2}{\Delta(x_1,x_2)} \\
&\times \Big[\overline{x}_1^2\, C_0([1-2]) - \overline{x}_1\,\overline{x}_2\, C_0([3-4]) - \overline{x}_1\,\overline{x}_2\, C_0([5-6]) - \overline{x}^2\, C_0([7-8]) + \overline{x}_1\,\overline{x}\, C_0([9-10])\Big], \\
I_{R;K} &= -\frac{1}{M^4}\int dS_2(x_1,x_2)\,\frac{x_2^2}{\Delta(x_1,x_2)} \\
&\times \Big\{-\frac{\overline{x}_1^3}{\overline{x}_2}\Big[C_{11}([1-2]) - C_{12}([1-2])\Big] - \overline{x}_1\,\overline{x}_2\Big[C_{11}([5-6]) + C_{12}([3-4])\Big] \\
&+ \overline{x}_1^2\Big[C_{11}([3-4]) + C_{12}([5-6])\Big] - \frac{\overline{x}^3}{\overline{x}_2}\Big[C_{11}([7-8]) - C_{12}([7-8])\Big] \\
&- \frac{\overline{x}_1\,\overline{x}^2}{\overline{x}_2}\, C_{12}([9-10]) + \overline{x}_1\,\overline{x}\, C_0([9-10]) - \frac{\overline{x}_1^2\,\overline{x}}{\overline{x}_2} C_{11}([9-10])\Big\},
\end{aligned} \tag{306}
$$

where $C_n([i-j]) = C_n(i) - C_n(j)$ and where $\overline{x}_i = 1 - x_i$, $\overline{x} = x_1 - x_2$. Furthermore we have

$$
\begin{aligned}
\Delta(x_1,x_2) &= \nu_x^2 - x_2\,(1-x_2)\,\mu_4^2 + x_2\,(1-x_1)\,(\mu_4^2 - \mu_6^2 + s_p), \\
\nu_x^2 &= -s_p\, x_1^2 + x_1\,(-s_p + \mu_1^2 - \mu_2^2) + x_2\,(\mu_3^2 - \mu_1^2) + \mu_2^2,
\end{aligned} \tag{307}
$$

where, according to III, we introduced $P^2 = -\,s_p\, M^2$, $\mu_i^2 = m_i^2/\mid P^2\mid$ and $\nu_j^2 = \mid p_j^2/P^2\mid$.

In this family we can show another example of ultraviolet divergent form factor in a situation where the corresponding scalar integral is convergent. Consider V_{114}^{K} defined in Eq.(244). Since the q_1 sub-loop diverges we expect a simple pole at $\epsilon = 0$. Let us define

$$
x_2\,(1-x_2)\,\chi_K(x_1,x_2,y_1+X\,y_3,y_2+X\,y_3,y_3) = \xi_K(\{x\},\{y\}) \equiv \xi_K(y_1,y_2,y_3), \tag{308}
$$

where χ_K is obtained from Eq.(225) by rescaling by $1/\mid P^2\mid$; ξ_K is a quadratic form in y_1, y_2, linear in y_3, with x-dependent coefficients. The procedure of extracting the ultraviolet pole (a subtraction, as introduced in III), followed by a mapping of the y_1, y_2 integration regions into the standard triangle $0 \le y_2 \le y_1 \le 1$, will introduce several new quadratic forms which will be enumerated as follows:

$$
\begin{aligned}
\xi(1-y_1,1-y_2) &= \xi_1, & \xi(1-X\,y_1,1-X\,y_2) &= \xi_2, \\
\xi(1-X\,y_1,1-y_2) &= \xi_3, & \xi(1-y_1,1-X\,y_2) &= \xi_4, \\
\xi(\overline{X}\,y_1,\overline{X}\,y_2) &= \xi_5, & \xi(1-X\,y_2,1-y_1) &= \xi_6, \\
\xi(1-y_2,1-X\,y_1) &= \xi_7, & \xi(1-X\,y_3\,y_1,1-X\,y_3\,y_2) &= \xi_8, \\
\xi(1-X\,y_1,\overline{X}\,y_2) &= \xi_9, & \xi(1-X\,y_3\,y_1,1-y_2) &= \xi_{10}, \\
\xi(1-X\,y_3\,y_2,1-y_1) &= \xi_{11}, & \xi(\overline{X}\,y_3\,y_2,\overline{X}\,y_3\,y_1) &= \xi_{12}, \\
\xi(1-y_1,\overline{X}\,y_2\,y_3) &= \xi_{13}, & \xi(1-X\,y_3\,y_2,\overline{X}\,y_3\,y_1) &= \xi_{14}, \\
\xi(1-X\,y_3\,y_1,\overline{X}\,y_3\,y_2) &= \xi_{15}, & \xi(1-y_2,\overline{X}\,y_3\,y_1) &= \xi_{16}, \\
\xi(1-X\,y_3\,y_1,1-X\,y_3\,y_2,y_3) &= \xi_{17}, & \xi(1-X\,y_3\,y_1,1-y_2,y_3) &= \xi_{18}, \\
\xi(1-X\,y_3\,y_1,\overline{X}\,y_3\,y_2,y_3) &= \xi_{19}, & \xi(1-y_1,\overline{X}\,y_3\,y_2,y_3) &= \xi_{20}, \\
\xi(1-y_2\,y_3\,X,1-y_1,y_3) &= \xi_{21}, & \xi(1-y_2\,y_3\,X,y_1\,y_3\,\overline{X},y_3) &= \xi_{22}, \\
\xi(1-y_2,y_1\,y_3\,\overline{X},y_3) &= \xi_{23}, & \xi(y_2\,y_3\,\overline{X},y_1\,y_3\,\overline{X},y_3) &= \xi_{24}, \\
\xi(1-y_2,1-y_1) &= \xi_{25}, & \xi(1-y_2,1-y_1,y_3) &= \xi_{26},
\end{aligned} \tag{309}
$$

where $\xi_K(y_1,y_2) \equiv \xi_K(y_1,y_2,0)$. We introduce new functions corresponding to well-defined integrals of the C-class:

$$
\int dS_2(y_1,y_2)\,\xi_l^{-1-\epsilon} = \mathcal{C}_0^0(l) - \frac{\epsilon}{2}\,\mathcal{C}_0^1(l) + \mathcal{O}\left(\epsilon^2\right), \qquad \mathcal{C}_{1i}^l(l) = \int dS_2(y_1,y_2)\,\xi_l^{-1}\,\ln y_i. \tag{310}
$$

All of them can be evaluated with the same algorithm described in Appendix E of III. Collecting all the ingredients we obtain

$$V^{K}_{114} = -\frac{1}{2}\left(\frac{\omega}{M^2}\right)^{\epsilon}\frac{\Gamma(1+\epsilon)}{M^4}\int dS_2(\{x\})\,x_2\left[V^{SP}_{114;K}\,\frac{1}{\epsilon} + V^{A}_{114;K} + \int_0^1 dy_3\,V^{B}_{114;K}\right], \tag{311}$$

$$V^{P}_{114;K} = 2\,\overline{x}_1\left[\mathcal{C}^0_{0}(3) + \mathcal{C}^0_{0}(6)\right] - 2\,\frac{\overline{x}_1^2}{\overline{x}_2}\,\mathcal{C}^0_{0}(2) + 2\,\frac{\overline{x}^2}{\overline{x}_2}\,\mathcal{C}^0_{0}(5), \tag{312}$$

$$\begin{aligned}
V^{A}_{114;K} &= -\frac{\overline{x}\,\overline{x}_1}{\overline{x}_2}\left[\mathcal{C}^l_{11}(9) - \mathcal{C}^l_{12}(9)\right] + \overline{x}_1\Big\{\left[\mathcal{C}^0_{0}(3) + \mathcal{C}^0_{0}(6)\right]\,\ln x_2\,(1-x_2)\\
&\quad - \mathcal{C}^1_{0}(3) - \mathcal{C}^1_{0}(6) + \mathcal{C}^l_{11}(3) + \mathcal{C}^l_{12}(6)\Big\} - \frac{\overline{x}_1^2}{\overline{x}_2}\left[\mathcal{C}^0_{0}(2)\,\ln x_2\,(1-x_2) - \mathcal{C}^1_{0}(2) + \mathcal{C}^l_{11}(2)\right]\\
&\quad + \frac{\overline{x}^2}{\overline{x}_2}\left[\mathcal{C}^0_{0}(5)\,\ln x_2\,(1-x_2) - \mathcal{C}^1_{0}(5) + \mathcal{C}^l_{12}(5)\right],\\
V^{B}_{114;K} &= -y_3\,x_2\,\frac{\overline{x}_1}{\overline{x}_2}\left[\mathcal{C}^0_{0}(18) + \mathcal{C}^0_{0}(21)\right] - y_3\,x_2\,\frac{\overline{x}}{\overline{x}_2}\left[\mathcal{C}^0_{0}(20) + \mathcal{C}^0_{0}(23)\right]\\
&\quad - y_3\,\frac{\overline{x}\,\overline{x}_1}{\overline{x}_2}\left[\mathcal{C}^0_{0}(14) + \mathcal{C}^0_{0}(15) - \mathcal{C}^0_{0}(19) - \mathcal{C}^0_{0}(22)\right] - y_3\,\frac{\overline{x}_1^2}{\overline{x}_2}\left[\mathcal{C}^0_{0}(8) - \mathcal{C}^0_{0}(17)\right]\\
&\quad + y_3^2\,x_2\,\frac{\overline{x}\,\overline{x}_1}{\overline{x}_2^2}\left[\mathcal{C}^0_{0}(19) + \mathcal{C}^0_{0}(22)\right] + y_3^2\,x_2\,\frac{\overline{x}_1^2}{\overline{x}_2^2}\,\mathcal{C}^0_{0}(17) + y_3^2\,x_2\,\frac{\overline{x}^2}{\overline{x}_2^2}\,\mathcal{C}^0_{0}(24)\\
&\quad + x_2\,\mathcal{C}^0_{0}(26) + \overline{x}_1\left[\mathcal{C}^0_{0}(10) + \mathcal{C}^0_{0}(11) - \mathcal{C}^0_{0}(18) - \mathcal{C}^0_{0}(21)\right] - \frac{\overline{x}_2}{y_3}\left[\mathcal{C}^0_{0}(25) - \mathcal{C}^0_{0}(26)\right]\\
&\quad + \overline{x}\left[\mathcal{C}^0_{0}(13) + \mathcal{C}^0_{0}(16) - \mathcal{C}^0_{0}(20) - \mathcal{C}^0_{0}(23)\right] - y_3\,\frac{\overline{x}^2}{\overline{x}_2}\left[\mathcal{C}^0_{0}(12) - \mathcal{C}^0_{0}(24)\right].
\end{aligned} \tag{313}$$

Note that in $V_{114\,;\,B;K}$ the C-functions have parameters which depend on x_1, x_2 and also on y_3.

The V^H-family is characterized by having ultraviolet finite components up to rank four tensors. Therefore, the techniques introduced in Sect. 10 of III can be transferred in an integral manner to all relevant form factors discussed in this article.

12 Conclusions

Any realistic calculation of physical observables in the framework of quantum field theory is remarkably more demanding than simply having at our disposal techniques to evaluate few special scalar diagrams. There are of course different strategies to compute complex diagrams but, to a large extent, they all amount to reducing a large number of integrals to some minimal set of master (irreducible) integrals.

As a starting procedure, one always saturates the Lorentz indices in the Green functions so that the numerator of the Feynman integrals contains powers of scalar products. The novelty in the analysis of two-loop vertices consists in the presence of so-called irreducible scalar products, namely, configurations in which the available propagators are not sufficient to algebraically simplify the numerator. Note that irreducible scalar products already occur in two-loop self-energies; there, however, the technique of reduction in sub-loops [7] alleviates their irreducibility (see our presentation in Section 5).

We showed that tensor integrals can be first of all decomposed into a combination of form factors, many of which can be reduced to scalar integrals (either of the same family or of families with a smaller number of propagators), while few irreducible integrals remain. It is then possible to relate these latter ones to generalized scalar integrals of the same family, i.e. integrals in shifted space-time dimensions and with non-canonical powers of the propagators. The number of these generalized scalar integrals can be further reduced using generalized recurrence relation techniques introduced by Tarasov in [27].

Alternatively, we developed our favorite strategy: following the findings of our work on one-loop multi-leg diagrams, we sought for a procedure where all integrals occurring in a realistic calculation can be written in a form analogous to Eq.(2). The practicality of this approach was strengthened in Section 11 by the explicit treatment of several form factors, paying particular attention to those cases where new or additional ultraviolet poles arise. In a line, we assembled the bases for extending a diagram generator to an evaluator of physical observables.

In our opinion, the optimal algorithm puts tensor integrals on the same footing as scalar ones and should not, therefore, introduce any multiplication of the tensor integrals by negative powers of Gram determinants. The numerical quality of tensor integrals should also not be worsened, as a consequence of the adopted reduction algorithm, by expressing them as linear combinations of master integrals; in this case, the kinematic coefficients have zeros corresponding to real singularities of the diagram, but their behavior around the singularity is always badly overestimated.

These shortcomings are not severe in the (almost) massless world of QED/QCD, but they turn into serious disadvantages in the massive world of the full-fledged Standard Model (SM). We found it more convenient to interpret irreducible configurations as integrals in the canonical $4-\epsilon$ dimensions with polynomials of Feynman parameters in the numerator; they can be computed – numerically – as well as the scalar ones. Several explicit examples were presented in Section 11.

Once we have reduced all obviously-reducible structures, we may as well compute all remaining quantities numerically. We must of course avoid situations where cancellations are expected: this may happen when the final result contains a very large number of terms, when apparent singularities are present (see Sect. D of III for a discussion) or when inherent gauge cancellations do not support a blind application of the procedure. We do not expect our approach to suffer from problems more severe than those encountered in other methods, but this remains to be fully tested in explicit two-loop applications. Comfortingly, our findings in numerical one-loop analysis (but also independent work [33]) support this claim.

In conclusion, we collected in one single place all the formulae needed to reduce fully massive tensor integrals, diagram-by-diagram up to three-point functions, to generalized scalar integrals. One may then choose how to proceed; for instance, using explicit integral representations for these functions and evaluating them with the same algorithms of smoothness (or with some of their generalizations) introduced in [5] for ordinary scalar functions.

Although we believe that there is no substitute for writing linearly, and that any article should be read linearly as well, we inserted several Appendices to be consulted as a reference.

The collection of results of this article contains all the ingredients needed to renormalize the SM (or any other renormalizable field theory) at the two-loop level, and to calculate the two-loop gauge boson complex poles as well as physical observables related to processes of the type $V(S) \to \overline{f} f$, the decay of vector or scalar particles into fermion–anti-fermion pairs. The use of projector techniques [20], augmented by the explicit reduction formulae that we collected, with the supplement of suitable integral representations for irreducible components, are the main tools to carry out the program.

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A Reduction for generalized one-loop functions

Generalized one-loop functions can be treated according to the BT-algorithm discussed in [3]. For $B_0(\alpha,\beta\,;\,p,m_1,m_2)$ one may use the results of Sect. 3 of [3], in particular Eqs. (23-24).

For the C-family there is full reducibility and, moreover, different scalar integrals can be related among each other and expressed in terms of standard scalar functions. The most convenient approach is based on the fact that all C-functions can be evaluated according to the BT-algorithm. We illustrate the procedure for functions of weight 4, where the weight is defined to be the sum of the (positive) powers in the propagators; all C functions can be written as

$$C[w=4] = \int dS_2\, P(x_1,x_2)\, V^{-2-\epsilon/2}(x_1,x_2), \quad P(x_1,x_2) = \sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm}\, x_1^n\, x_2^m, \tag{314}$$

where $w = \alpha_1+\alpha_2+\alpha_3$ and where the polynomial P depends on the specific case under consideration. For standard ($w=3$) form factors (from C_{11} to C_{24}) the corresponding polynomials are given in Eq. (41) of [3]. For scalar functions of weight 4 the P are $1-x_1$ for $C_0(2,1,1)$, x_1-x_2 for $C_0(1,2,1)$ and x_2 for $C_0(1,1,2)$.

Higher weights can be evaluated recursively, e.g.

$$\begin{aligned}
C[w=4] &= \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{a_{nm}}{(2+\epsilon)\,B_3}\, C_{nm}[w=4],\\
C_{nm}[w=4] &= \int dS_2\; V^{-1-\epsilon/2}(x_1,x_2)\,\Big[n\,X_1\,x_1^{n-1}\,x_2^m + m\,X_2\,x_1^n\,x_2^{m-1} + (\epsilon-n-m)\,x_1^n\,x_2^m\Big]\\
&\quad + \int dC_1\,\Big[(1-X_1)\,x_1^m\,V^{-1-\epsilon/2}(1,x_1) + (X_1-X_2)\,x_1^{n+m}\,V^{-1-\epsilon/2}(x_1,x_1)\\
&\quad + \delta_{m,0}\,X_2\,V^{-1-\epsilon/2}(x_1,0)\Big],
\end{aligned} \tag{315}$$

where, with the definition of Eq.(19), the quadratic form V is

$$\begin{aligned}
V(x_1,x_2) &= x^t\,G\,x + 2\,K^t\,x + L, \qquad G_{ij} = -\,p_i\cdot p_j, \quad L = m_1^2,\\
K_1 &= \frac{1}{2}\,(p_1^2+m_2^2-m_1^2), \quad K_2 = \frac{1}{2}\,(P^2-p_1^2+m_3^2-m_2^2),
\end{aligned} \tag{316}$$

with $P = p_1+p_2$. Furthermore, $B_3 = L - K^t\,G^{-1}\,K$ and $X = -\,G^{-1}\,K$. For $w=3$ we finally have

$$C_{nm}[w=3] = C^0_{mn} - \frac{1}{2}\int dC_1\, C^1_{mn} - \frac{1}{2}\int dS_2\, x_1^{n-1}\,x_2^{m-1} C^2_{mn}\Big], \tag{317}$$

where the coefficients are

$$\begin{aligned}
C^0_{mn} &= \frac{1}{(2+n+m)\,(1+m)},\\
C^1_{mn} &= (X_1-X_2)\,x_1^{n+m}\,\ln V(x_1,x_1) + \delta_{m,0}\,X_2\,x_1^n\,\ln V(x_1,0) + (1-X_1)\,x_1^m\,\ln V(1,x_1),\\
C^2_{mn} &= (m\,X_2\,x_1 - (2+n+m)\,x_1\,x_2 + n\,X_1\,x_2)\,\ln V(x_1,x_2).
\end{aligned} \tag{318}$$

D-family functions of weight 5 can be reduced recursively with three iterations of the BT-algorithm; see Sect. (6.2) of [3] for a discussion, in particular Eqs. (140-142). These functions are not needed in this paper.

B Summary of the results for the reduction of three-point functions

In this Section we present a summary of the results obtained for the reduction of two-loop three-point functions and derived in Sections 9.1–9.6. Tensor integrals are defined by having powers of momenta in the numerator; they are further decomposed into form factors $\otimes$ tensor structures and, for completeness, the full collection of results is presented for the form factors as well as for tensor integrals with saturated indices; the latter are perhaps the most important objects when one computes physical amplitudes in the framework of the projector techniques introduced in Section 4.

The presentation is organized through a series of concatenated formulae that can be easily coded with any of the well-known packages for symbolic manipulation; the formulae below can thus be used as they stand or they can be used recursively. Each object of the list contains scalar functions or form factors corresponding to tensors of lower rank and with fewer propagators that can be found earlier in the list and, if needed, the procedure can be iterated until the chain of reductions stops with a fully scalarized expression.

Formulae for ordinary scalar vertex functions can be found in III where, however, the alphameric convention was not yet used and therefore the correspondence is based on Eq.(25). In particular, $V_0^{E} \equiv V_0^{121}$ in Sect. 5.1, $V_0^{I} \equiv V_0^{131}$ in Sect. 6.1, $V_0^{G} \equiv V_0^{221}$ in Sect. 7.1, $V_0^{M} \equiv V_0^{141}$ in Sects. 8.1 − 8.2, $V_0^{K} \equiv V_0^{231}$ in Sects. 9.1 − 9.2 and $V_0^{H} \equiv V_0^{222}$ in Sect. 10.4 of III. Additional material, with the extension to generalized scalar functions, is presented in this paper in Sects. 11.1 − 11.4. Finally, reduction of one-loop generalized form factors has been discussed in Appendix A.

Additional notation, relevant for this Appendix, was given in the Introduction but is repeated here: we denote by G the Gram matrix arising in the context of a vertex function and use

$$G_{ij} = p_i \cdot p_j, \quad D = \det G = p_1^2\, p_2^2 - (p_1 \cdot p_2)^2, \quad D_1 = p_1^2\, p_2^2, \quad D_2 = p_{12}\, p_2^2, \quad D_3 = p_{12}\, p_1^2. \tag{319}$$

Before presenting the list of results we would like to discuss one specific example. Consider a rank two tensor, e.g. from Section 9.5, where all indices are saturated with external momenta:

$$\begin{aligned} V^{K}(0\,|\,p_1, p_1) = \frac{1}{2}\Big\{ &-l_{145}\Big[V^{K}_{21}\, p_1^2 + V^{K}_{22}\, p_{12}\Big] - V^{G}_{21}(P, P, p_1, \{m\}_{12365})\, p_1^2 \\ &+ p_1 \cdot P\Big[V^{G}_{21}(P, P, 0, \{m\}_{12364}) - V^{G}_{22}(P, P, 0, \{m\}_{12364})\Big]\Big\}. \end{aligned} \tag{320}$$

After the first step in the reduction there is no Gram determinant but the latter may still be hidden in form factors corresponding to tensors of lower rank. As a matter of fact, we may iterate the procedure and consider

$$\begin{aligned} V^{K}_{21}\, p_1^2 + V^{K}_{22}\, p_{12} &= V^{K}(0\,|\,p_1), \\ p_1 \cdot P\Big[V^{G}_{21}(P, P, 0, \{m\}_{12364}) - V^{G}_{22}(P, P, 0, \{m\}_{12364})\Big] &= V^{G}(0\,|\,p_1\,;\, P, P, 0, \{m\}_{12364}), \\ V^{G}_{21}(P, P, p_1, \{m\}_{12365}). \end{aligned} \tag{321}$$

A reduction, which is again free from Gram determinants, can be applied to the first term in Eq.(321); however, further scalarization for the last two can only be performed if Gram determinants do not pose a problem, as in proving WST identities; otherwise the reduction chain for these terms should stop and their evaluation will follow according to the corresponding defining representation (note that $V^{G}(0\,|\,P\,;\, P, P, 0, \{m\}_{12364})$ is instead fully reducible). Alternatively, the term can be further reduced with generalized recurrence relations which, however, introduce additional kinematic coefficients, with the appearance of (physical) singularities etc, etc.

In summarizing the whole set of results we adopt the following convention: the list of arguments of tensor integrals in a given class is suppressed when we present their reduction; therefore

$$\begin{array}{ll} V^{E}_{J} \equiv V^{E}_{J}(p_2, P, \{m\}_{1234}), & V^{I}_{J} \equiv V^{I}_{J}(p_1, P, \{m\}_{12345}), \\ V^{M}_{J} \equiv V^{M}_{J}(p_1, P, \{m\}_{12345}), & V^{G}_{J} \equiv V^{G}_{J}(p_1, p_1, P, \{m\}_{12345}), \\ V^{K}_{J} \equiv V^{K}_{J}(P, p_1, P, \{m\}_{123456}), & V^{H}_{J} \equiv V^{H}_{J}(-p_2, p_1, -p_2, -p_1, \{m\}_{123456}), \end{array}$$

where J denotes a generic form factor in the family and where, for the M family we always assume $m_6 = m_3$. We also report, for completeness the definitions of all form factors occurring in our paper:

$$V^{J}(\mu\,|\,0\,;\,\cdots) = \sum_{i=1,2} V^{J}_{1i}\, p_{i\mu}, \quad V^{J}(0\,|\,\mu\,;\,\cdots) = \sum_{i=1,2} V^{J}_{2i}\, p_{i\mu}, \qquad J = E, I, M, G, K, H,$$

$$
\begin{aligned}
V^E(\mu,\nu\,|\,0\,;\cdots) &= V^E_{111}\,p_{1\mu}\,p_{1\nu} + V^E_{112}\,p_{2\mu}\,p_{2\nu} + V^E_{113}\,\{p_1\,p_2\}_{\mu\nu} + V^E_{114}\,\delta_{\mu\nu}, \quad \text{etc},\\
V^I(0\,|\,\mu,\nu\,;\cdots) &= V^I_{221}\,p_{1\mu}\,p_{1\nu} + V^I_{222}\,p_{2\mu}\,p_{2\nu} + V^I_{223}\,\{p_1p_2\}_{\mu\nu} + V^I_{224}\,\delta_{\mu\nu}, \quad \text{etc},\\
V^M(0\,|\,\mu,\nu\,;\cdots) &= V^M_{221}\,p_{1\mu}\,p_{1\nu} + V^M_{222}\,p_{2\mu}\,p_{2\nu} + V^M_{223}\,\{p_1p_2\}_{\mu\nu} + V^M_{224}\,\delta_{\mu\nu}, \quad \text{etc},\\
V^G(0\,|\,\mu,\nu\,;\cdots) &= V^G_{221}\,p_{1\mu}\,p_{1\nu} + V^G_{222}\,p_{2\mu}\,p_{2\nu} + V^G_{223}\,\{p_1p_2\}_{\mu\nu} + V^G_{224}\,\delta_{\mu\nu}\\
V^G(\mu\,|\,\nu\,;\cdots) &= V^G_{121}\,p_{1\mu}\,p_{1\nu} + V^G_{122}\,p_{2\mu}\,p_{2\nu} + V^G_{123}\,p_{1\mu}\,p_{2\nu} + V^G_{125}\,p_{1\nu}\,p_{2\mu} + V^G_{124}\,\delta_{\mu\nu},\\
V^G(\mu,\nu\,|\,0\,;\cdots) &= V^G_{111}\,p_{1\mu}\,p_{1\nu} + V^G_{112}\,p_{2\mu}\,p_{2\nu} + V^G_{113}\,\{p_1p_2\}_{\mu\nu} + V^G_{114}\,\delta_{\mu\nu},\\
V^K(0\,|\,\mu,\nu\,;\cdots) &= V^K_{221}\,p_{1\mu}\,p_{1\nu} + V^K_{222}\,p_{2\mu}\,p_{2\nu} + V^K_{223}\,\{p_1\,p_2\}_{\mu\nu} + V^K_{224}\,\delta_{\mu\nu},\\
V^K(\mu\,|\,\nu\,;\cdots) &= V^K_{121}\,p_{1\mu}\,p_{1\nu} + V^K_{122}\,p_{2\mu}\,p_{2\nu} + V^K_{123}\,p_{1\mu}\,p_{2\nu} + V^K_{125}\,p_{1\nu}\,p_{2\mu} + V^K_{124}\,\delta_{\mu\nu},\\
V^K(\mu,\nu\,|\,0\,;\cdots) &= V^K_{111}\,p_{1\mu}\,p_{1\nu} + V^K_{112}\,p_{2\mu}\,p_{2\nu} + V^K_{113}\,\{p_1p_2\}_{\mu\nu} + V^K_{114}\,\delta_{\mu\nu},\\
V^H(0\,|\,\mu,\nu\,;\cdots) &= V^H_{221}\,p_{1\mu}\,p_{1\nu} + V^H_{222}\,p_{2\mu}\,p_{2\nu} + V^H_{223}\,\{p_1p_2\}_{\mu\nu} + V^H_{224}\,\delta_{\mu\nu},\\
V^H(\mu,\nu\,|\,0;\cdots) &= V^H_{111}\,p_{1\mu}\,p_{1\nu} + V^H_{112}\,p_{2\mu}\,p_{2\nu} + V^H_{113}\,\{p_1p_2\}_{\mu\nu} + V^H_{114}\,\delta_{\mu\nu},\\
V^H(\mu\,|\,\nu;\cdots) &= V^H_{E}\,p_{1\mu}\,p_{1\nu} + V^H_{122}\,p_{2\mu}\,p_{2\nu} + V^H_{123}\,\{p_1\,p_2\}_{\mu\nu} + V^H_{124}\,\delta_{\mu\nu},\\
V^M(\mu,\nu,\alpha\,|\,0;\cdots) &= V^M_{1111}\,\{\delta\,p_1\}_{\mu\nu\alpha} + V^M_{1112}\,\{\delta\,p_2\}_{\mu\nu\alpha} + V^M_{1113}\,\{p_1p_1p_2\}_{\mu\nu\alpha} + V^M_{1114}\,\{p_2p_2p_1\}_{\mu\nu\alpha}\\
&+ V^M_{1115}\,p_{1\mu}\,p_{1\nu}\,p_{1\alpha} + V^M_{1116}\,p_{2\mu}\,p_{2\nu}\,p_{2\alpha}, \quad \text{etc},\\
V^M(0\,|\,\mu,\nu,\alpha;\cdots) &= V^M_{2221}\,\{\delta\,p_1\}_{\mu\nu\alpha} + V^M_{2222}\,\{\delta\,p_2\}_{\mu\nu\alpha} + V^M_{2223}\,\{p_1p_1p_2\}_{\mu\nu\alpha} + V^M_{2224}\,\{p_2p_2p_1\}_{\mu\nu\alpha}\\
&+ V^M_{2225}\,p_{1\mu}\,p_{1\nu}\,p_{1\alpha} + V^M_{2226}\,p_{2\mu}\,p_{2\nu}\,p_{2\alpha}, \quad \text{etc},\\
V^M(\mu\,|\,\nu,\alpha;\cdots) &= V^M_{1221}\,\{\delta\,p_1\}_{\mu\nu\alpha} + V^M_{1222}\,\{\delta\,p_2\}_{\mu\nu\alpha} + V^M_{1223}\,\{p_1p_1p_2\}_{\mu\nu\alpha} + V^M_{1224}\,\{p_2p_2p_1\}_{\mu\nu\alpha}\\
&+ V^M_{1225}\,p_{1\mu}\,p_{1\nu}\,p_{1\alpha} + V^M_{1226}\,p_{2\mu}\,p_{2\nu}\,p_{2\alpha},\\
V^M(\mu,\nu\,|\,\alpha;\cdots) &= V^M_{1121}\,\{\delta\,p_1\}_{\mu\nu\alpha} + V^M_{1122}\,\{\delta\,p_2\}_{\mu\nu\alpha} + V^M_{1123}\,\{p_1p_1p_2\}_{\mu\nu\alpha} + V^M_{1124}\,\{p_2p_2p_1\}_{\mu\nu\alpha}\\
&+ V^M_{1125}\,p_{1\mu}\,p_{1\nu}\,p_{1\alpha} + V^M_{1126}\,p_{2\mu}\,p_{2\nu}\,p_{2\alpha} + V^M_{1127}\,\{\delta\,p_1\}_{\mu\nu\,|\,\alpha} + V^M_{1128}\,\{\delta\,p_2\}_{\mu\nu\,|\,\alpha},\\
V^K(\mu\,|\,\nu,\alpha;\cdots) &= V^K_{1221}\,\{\delta\,p_1\}_{\nu\alpha\,|\,\mu} + V^K_{1222}\,\{\delta\,p_2\}_{\nu\alpha\,|\,\mu} + V^K_{1223}\,\delta_{\nu\alpha}p_{1\mu} + V^K_{1224}\,\delta_{\nu\alpha}p_{2\mu}\\
&+ V^K_{1225}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu} + V^K_{1226}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^K_{1227}\,\{p_1\,p_1\,p_2\}_{\alpha\nu\,|\,\mu}\\
&+ V^K_{1228}\,\{p_2\,p_2\,p_1\}_{\nu\alpha\,|\,\mu} + V^K_{1229}\,p_{1\alpha}\,p_{1\nu}\,p_{2\mu} + V^K_{12210}\,p_{1\mu}\,p_{2\nu}\,p_{2\alpha},\\
V^K(\mu,\nu\,|\,\alpha;\cdots) &= V^K_{1121}\,\{\delta\,p_1\}_{\mu\nu\,|\,\alpha} + V^K_{1122}\,\{\delta\,p_2\}_{\mu\nu\,|\,\alpha} + V^K_{1123}\,\delta_{\mu\nu}p_{1\alpha} + V^K_{1124}\,\delta_{\mu\nu}p_{2\alpha}\\
&+ V^K_{1125}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu} + V^K_{1126}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^K_{1127}\,\{p_1p_1p_2\}_{\mu\nu\,|\,\alpha}\\
&+ V^K_{1128}\,\{p_2p_2p_1\}_{\nu\mu\,|\,\alpha} + V^K_{1129}\,p_{1\mu}\,p_{1\nu}\,p_{2\alpha} + V^K_{11210}\,p_{1\alpha}\,p_{2\mu}\,p_{2\nu},\\
V^H(\mu\,|\,\nu,\alpha;\cdots) &= V^H_{1221}\,\{\delta\,p_1\}_{\nu\alpha\,|\,\mu} + V^H_{1222}\,\{\delta\,p_2\}_{\nu\alpha\,|\,\mu} + V^H_{1223}\,\delta_{\nu\alpha}p_{1\mu} + V^H_{1224}\,\delta_{\nu\alpha}p_{2\mu} + V^H_{1225}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu}\\
&+ V^H_{1226}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^H_{1227}\,\{p_1\,p_1\,p_2\}_{\mu\nu\alpha} + V^H_{1228}\,\{p_1\,p_2\,p_2\}_{\mu\nu\alpha},\\
V^H(\mu,\nu\,|\,\alpha;\cdots) &= V^H_{1121}\,\{\delta\,p_1\}_{\mu\nu\,|\,\alpha} + V^H_{1122}\,\{\delta\,p_2\}_{\mu\nu\,|\,\alpha} + V^H_{1123}\,\delta_{\mu\nu}p_{1\alpha} + V^H_{1124}\,\delta_{\mu\nu}p_{2\alpha} + V^H_{1125}\,p_{1\alpha}\,p_{1\mu}\,p_{1\nu}\\
&+ V^H_{1126}\,p_{2\alpha}\,p_{2\mu}\,p_{2\nu} + V^H_{1127}\,\{p_1\,p_1\,p_2\}_{\mu\nu\alpha} + V^H_{1128}\,\{p_1\,p_2\,p_2\}_{\mu\nu\alpha}. \qquad (322)
\end{aligned}
$$

We are now ready to summarize the results; tags were introduced to facilitate the search of the various items, for instance results related to rank two tensor integrals of the kl group ($kl = \{11, 12, 22\}$) in the J family ($J = E, I, M, G, K$ and H) are to be searched under the tag $\mathbf{V^J_{kli}}$.

B.1 $V^E(p_2, P, \{m\}_{1234})$ family

$\boxed{\mathbf{V^E_{ij}}}$ Results were derived in Section 9.1.1. Referring to Eq.(126), for vector integrals we have

$$
\begin{aligned}
V^E_{1i} &= \frac{1}{2}\Big[V^E_{2i} + m^2_{21}\,V^I_{2i}(p_2, P, \{m\}_{12}, 0, \{m\}_{34}) - C_{1i}(p_2, p_1, 0, \{m\}_{34})\,A_0([m_2, m_1])\Big],\\
V^E_{21} &= -\omega^2\Big[V_E^{2|1,2|1} + V_E^{1|1,2|2}\Big]\Big|_{n=6-\epsilon}, \quad V^E_{22} = -\omega^2\Big[V_E^{2|2,1|1} + V_E^{2|1,2|1} + V_E^{1|2,1|2} + V_E^{1|1,2|2}\Big]\Big|_{n=6-\epsilon} \qquad (323)
\end{aligned}
$$

$\boxed{\mathbf{V^E_{22i}}}$ For tensor integrals (see Section 9.1.2) we introduce a vector U^E_{22} with components

$$
U^E_{22\,;\,1} = \frac{1}{2}\Big[-V^E_{22}\,(2\,p_{12} + l_{134}) + S^A_0(P, \{m\}_{124}) - S^A_0(p_2, \{m\}_{123})
$$

$$
\begin{aligned}
&\quad + S^{A}_{2}(P,\{m\}_{124}) - S^{A}_{2}(p_2,\{m\}_{123})\Big], \\
U^{E}_{22\,;\,2} &= -V^{E}_{0}\,(p_2^2+m_3^2) - V^{E}_{21}\,(p_{12} - \frac{1}{2}\,l_{134}) - 2\,V^{E}_{22}\,p_2^2 - (n-1)\,V^{E}_{224} \\
&\quad + \frac{1}{2}\,S^{A}_{0}(P,\{m\}_{124}) - \frac{1}{2}\,S^{E}_{2}(P,\{m\}_{124}),
\end{aligned}
\tag{324}
$$

then one obtains

$$
\begin{pmatrix} V^{E}_{223} \\ V^{E}_{222} \end{pmatrix} = G^{-1}\,U^{E}_{22}, \qquad p_1^2\,V^{E}_{221} = -\,p_{12}\,V^{E}_{223} - V^{E}_{224} + U^{E}_{22\,;\,2},
\tag{325}
$$

with a generalized function

$$
V^{E}_{224} = \frac{1}{2}\,\omega^2\,\Big[V^{2|1,1|1}_{E} + V^{1|1,1|2}_{E}\Big]\Big|_{n=6-\epsilon}.
\tag{326}
$$

For tensors with saturated indices we obtain

$$
\begin{aligned}
V^{E}(0\,|\,\mu,\mu) &= -V^{E}_{0}\,(p_2^2+m_3^2) - 2\,V^{E}_{22}\,p_2^2 - 2\,V^{E}_{21}\,p_1^2 + S^{A}_{0}(P,\{m\}_{124}) \\
V^{E}(0\,|\,p_1,p_1) &= -(p_{12}\,V^{E}_{22} + p_1^2\,V^{E}_{21})\,(p_{12} + \frac{1}{2}\,l_{134}) + \frac{1}{2}\,p_1\cdot P\,\Big[S^{A}_{0}(P,\{m\}_{124}) + S^{A}_{2}(P,\{m\}_{124})\Big] \\
&\quad - \frac{1}{2}\,p_{12}\,\Big[S^{A}_{0}(p_2,\{m\}_{123}) + S^{A}_{2}(p_2,\{m\}_{123})\Big], \\
p_1^2\,V^{E}(0\,|\,p_2,p_2) &= -V^{E}_{224}\,(n-2)\,D - V^{E}_{22}\,p_2^2\,(2\,D + \frac{1}{2}\,p_{12}\,l_{134} + p_{12}^2) \\
&\quad - V^{E}_{0}\,(p_2^2+m_3^2)\,D - V^{E}_{21}\,\Big[D\,(p_{12} - \frac{1}{2}\,l_{134}) + p_{12}^2\,(p_{12} + \frac{1}{2}\,l_{134})\Big] \\
&\quad + \frac{1}{2}\,S^{A}_{0}(P,\{m\}_{124})\,(D + D_2 + p_{12}^2) - \frac{1}{2}\,S^{A}_{0}(p_2,\{m\}_{123})\,D_2 \\
&\quad - \frac{1}{2}\,S^{A}_{2}(P,\{m\}_{124})\,(D - D_2 - p_{12}^2) - \frac{1}{2}\,S^{A}_{2}(p_2,\{m\}_{123})\,D_2, \\
V^{E}(0\,|\,p_1,p_2) &= -(V^{E}_{22}\,p_2^2 + V^{E}_{21}\,p_{12})\,(\frac{1}{2}\,l_{134} + p_{12}) + \frac{1}{2}\,p_2\cdot P\,\Big[S^{A}_{0}(P,\{m\}_{124}) + S^{A}_{2}(P,\{m\}_{124})\Big] \\
&\quad - \frac{1}{2}\,p_2^2\,\Big[S^{A}_{0}(p_2,\{m\}_{123}) + S^{A}_{2}(p_2,\{m\}_{123})\Big].
\end{aligned}
\tag{327}
$$

$\boxed{\mathbf{V^{E}_{12i}}}$ Furthermore we obtain

$$
V^{E}_{12i} = \frac{1}{2}\,\Big[V^{E}_{22i} + m_{21}^2\,V^{I}_{22i}(p_2,P,\{m\}_{12},0,\{m\}_{34}) + \,C_{2i}(p_2,p_1,0,\{m\}_{34})\,A_0([m_2,m_1])\Big].
\tag{328}
$$

$\boxed{\mathbf{V^{E}_{11i}}}$ Finally, for $i<4$ we have

$$
\begin{aligned}
4\,(n-1)\,V^{E}_{11i} &= n\,V^{E}_{22i} + n\,m_{12}^4\,V^{M}_{22i}(p_1,P,\{m\}_{12},0,\{m\}_{34},0) \\
&\quad + 2\,(n\,m_{21}^2 + 2\,m_1^2)\,V^{I}_{22i}(p_2,P,\{m\}_{12},0,\{m\}_{34}) \\
&\quad - n\,A_0(m_1)\,\Big[m_{12}^2\,C_{2i}(2,1,1\,;\,p_2,p_1,0,\{m\}_{34}) - C_{2i}(p_2,p_1,0,\{m\}_{34})\Big] \\
&\quad - A_0(m_2)\,\Big[(3\,n-4)\,C_{2i}(p_2,p_1,0,\{m\}_{34}) + n\,m_{21}^2\,C_{2i}(2,1,1\,;\,p_2,p_1,0,\{m\}_{34})\Big],
\end{aligned}
\tag{329}
$$

while, for $i=4$ it follows that

$$
\begin{aligned}
4\,(n-1)\,V^{E}_{114} &= n\,V^{E}_{224} - 2\,(m_1^2+m_2^2)\,V^{E}_{0} - V^{E}(0|\mu,\mu) + n\,m_{12}^4\,V^{M}_{224}(p_1,P,\{m\}_{12},0,\{m\}_{34},0) \\
&\quad + 2\,(n\,m_{21}^2 + 2\,m_1^2)\,V^{I}_{224}(p_2,P,\{m\}_{12},0,\{m\}_{34}) - m_{12}^4\,V^{I}_{0}(p_2,P,\{m\}_{12},0,\{m\}_{34}) \\
&\quad - n\,A_0(m_1)\,\Big[m_{12}^2\,C_{24}(2,1,1\,;\,p_2,p_1,0,\{m\}_{34}) - C_{24}(p_2,p_1,0,\{m\}_{34})\Big] \\
&\quad - A_0(m_2)\,\Big[(3\,n-4)\,C_{24}(p_2,p_1,0,\{m\}_{34}) + n\,m_{12}^2\,C_{24}(2,1,1\,;\,p_2,p_1,0,\{m\}_{34})\Big]
\end{aligned}
$$

$$- A_0(m_1)\Big[m_{21}^2\,C_0(p_2,p_1,0,\{m\}_{34}) + B_0(\,p_1,\{m\}_{34})\Big]$$
$$- A_0(m_2)\Big[m_{12}^2\,C_0(p_2,p_1,0,\{m\}_{34}) + B_0(\,p_1,\{m\}_{34})\Big]. \qquad (330)$$

Eq.(330) requires some of the results corresponding to the V^I family: they are collected in Section B.2.

B.2 $V^I(p_1, P, \{m\}_{12345})$ family

$\boxed{\mathbf{V^I_{2i}}}$ Results were derived in Section 9.2.1. Introduce a vector U_2^I with components

$$U^I_{2\,;\,1} = \frac{1}{2}\Big[-l_{134}V^I_0 - V^E_0(p_1,P,\{m\}_{1245}) + V^E_0(0,P,\{m\}_{1235})\Big],$$
$$U^I_{2\,;\,2} = \frac{1}{2}\Big[(l_{154}{-}P^2)V^I_0 + V^E_0(0,p_1,\{m\}_{1234}) - V^E_0(0,P,\{m\}_{1235})\Big]; \qquad (331)$$

we obtain the following result:

$$\begin{pmatrix} V^I_{21} \\ V^I_{22} \end{pmatrix} = G^{-1}\,U^I_2, \qquad V^I(0\,|\,p_1) = U^I_{2\,;\,1}, \qquad V^I(0\,|\,p_2) = U^I_{2\,;\,2}. \qquad (332)$$

$\boxed{\mathbf{V^I_{1i}}}$ Furthermore we have

$$V^I_{1i}(p_1,P,\{m\}_{12345}) = \frac{1}{2}\,\frac{m_{123}^2}{m_3^2}\,V^I_{2i}(p_1,P,\{m\}_{12345}) - \frac{1}{2}\,\frac{m_{12}^2}{m_3^2}\,V^I_{2i}(p_1,P,\{m\}_{12},0,\{m\}_{45})$$
$$- \frac{1}{2m_3^2}\,A_0([m_1,m_2])\Big\{\delta_{i1}\sum_{j=1,2}\Big[C_{1j}(p_1,p_2,\{m\}_{345}) - C_{1j}(p_1,p_2,0,\{m\}_{45})\Big]$$
$$+ \delta_{i2}\Big[C_{12}(p_1,p_2,\{m\}_{345}) - C_{12}(p_1,p_2,0,\{m\}_{45})\Big]\Big\}. \qquad (333)$$

$\boxed{\mathbf{V^I_{22i}}}$ For rank two tensors (see Section 9.2.2) we introduce a vector U^I_{22} with components

$$U^I_{22\,;\,1} = \frac{1}{2}\Big[-l_{134}\,V^I_{21} - V^E_{21}(p_1,P,\{m\}_{1245}) + V^E_{21}(0,P,\{m\}_{1235})\Big],$$
$$U^I_{22\,;\,2} = -\,V^I_{224} + \frac{1}{2}\,(p_1^2 - l_{P45})\,V^I_{22} - \frac{1}{2}\,V^E_{21}(0,P,\{m\}_{1235})\,. \qquad (334)$$

We obtain the following result:

$$\begin{pmatrix} V^I_{223} \\ V^I_{222} \end{pmatrix} = G^{-1}\,U^I_{22}. \qquad (335)$$

Introduce a vector W^I_{22} with components

$$W^I_{22\,;\,1} = -\,V^I_{224} - \frac{1}{2}\,l_{134}\,V^I_{22} - \frac{1}{2}\,V^E_{21}(p_1,P,\{m\}_{1245}) + \frac{1}{2}\,V^E_{22}(0,P,\{m\}_{1235}),$$
$$W^I_{22\,;\,2} = \frac{1}{2}\Big[(p_1^2 - l_{P45})\,V^I_{21} - V^E_{21}(0,P,\{m\}_{1235}) + V^E_{21}(0,p_1,\{m\}_{1234}\Big]. \qquad (336)$$

We obtain the following result:

$$\begin{pmatrix} V^I_{221} \\ V^I_{223} \end{pmatrix} = G^{-1}\,W^I_{22}. \qquad (337)$$

Furthermore, we get

$$(2-n)\,V^I_{224} = V^I_0\,m_3^2 - V^E_0(p_1,P,\{m\}_{1245}) - \frac{1}{2}\Big[V^I_{22}\,l_{P35} + V^E_{21}(p_1,P,\{m\}_{1245})$$
$$+ V^E_{21}(0,P,\{m\}_{1235}) - V^E_{22}(0,P,\{m\}_{1235})\Big]. \qquad (338)$$

For the corresponding tensors with saturated indices we obtain

$$
\begin{aligned}
V^{I}(0\,|\,\mu,\mu) &= -V^{I}_{0}\, m_3^2 + V^{E}_{0}(p_1,P,\{m\}_{1245}),\\
V^{I}(0\,|\,p_1,p_1) &= \frac{1}{2}\Big[-l_{134}\,(V^{I}_{21}\,p_{12} + V^{I}_{22}\,p_1^2)\\
&\quad - V^{E}_{21}(p_1,P,\{m\}_{1245})\,p_1\cdot P + V^{E}_{21}(0,P,\{m\}_{1235})\,p_{12} + V^{E}_{22}(0,P,\{m\}_{1235})\,p_1^2\Big],\\
p_1^2\,V^{I}(0\,|\,p_2,p_2) &= \frac{1}{2}\Big[-V^{I}_{21}\,D_2\,l_{134} + V^{I}_{22}\,(D\,p_1^2 + D\,l_{134} - D\,l_{P45} - D_1\,l_{134}) - V^{E}_{21}(p_1,P,\{m\}_{1245})\,(p_{12}^2 + D_2)\\
&\quad + V^{E}_{21}(0,P,\{m\}_{1235})\,(-D + D_2) + V^{E}_{22}(0,P,\{m\}_{1235})\,p_{12}^2)\Big],\\
V^{I}(0\,|\,p_1,p_2) &= \frac{1}{2}\Big[-l_{134}\,(V^{I}_{21}\,p_2^2 + V^{I}_{22}\,p_{12})\\
&\quad - V^{E}_{21}(p_1,P,\{m\}_{1245})\,p_2\cdot P + V^{E}_{21}(0,P,\{m\}_{1235})\,p_2^2 + V^{E}_{22}(0,P,\{m\}_{1235})\,p_{12}\Big].
\end{aligned}
\tag{339}
$$

$\boxed{\mathbf{V^{I}_{12i}}}$ The form factors corresponding to the 12 and 22 groups are related by

$$
\begin{aligned}
V^{I}_{12i}(p_1,P,\{m\}_{12345}) &= \frac{m_{123}^2}{2m_3^2}\,V^{I}_{22i}(p_1,P,\{m\}_{12345}) + \frac{m_{21}^2}{2m_3^2}\,V^{I}_{22i}(p_1,P,\{m\}_{12},0,\{m\}_{45})\\
&\quad + \Delta V^{I}_{12i}(p_1,P,\{m\}_{12345}),
\end{aligned}
\tag{340}
$$

$$
\Delta V^{I}_{12i} = -\frac{1}{2m_3^2}\,A_0([m_1,m_2])\Big[C_{2i}(p_1,p_2,\{m\}_{345}) - C_{2i}(p_1,p_2,0,\{m\}_{45})\Big], \quad i = 1\cdots 4.
\tag{341}
$$

$\boxed{\mathbf{V^{I}_{11i}}}$ The form factors corresponding to the 11 and 22 groups are related by

$$
\begin{aligned}
4\,(n-1)\,m_3^2\,V^{I}_{11i} &= m_3^2\,(2\,n\,m_{12}^2 + n\,m_3^2 - 4\,m_1^2)\,V^{I}_{22i}(p_1,P,\{m\}_{12345})\\
&\quad - m_3^2\Big[(n-4)\,m_1^2 - 2\,n\,m_2^2\Big]\,V^{I}_{22i}(p_1,P,\{m\}_{12},0,\{m\}_{45})\\
&\quad + n\,m_3^2\,m_{12}^4\Big[V^{M}_{22i}(p_1,P,\{m\}_{12},0,\{m\}_{45},0) - V^{M}_{22i}(p_1,P,\{m\}_{12345},0)\Big]\\
&\quad - n\,m_3^2\,A_0(m_1)\Big[C_{2i}\,(p_1,p_2,\{m\}_{345}) - C_{2i}\,(p_1,p_2,0,\{m\}_{45})\Big]\\
&\quad - n\,m_{12}^2\,A_0([m_1,m_2])\Big[m_3^2\,C_{2i}(2,1,1\,;\,p_1,p_2,0,\{m\}_{45})\\
&\quad - C_{2i}\,(p_1,p_2,0,\{m\}_{45}) + C_{2i}\,(p_1,p_2,\{m\}_{345})\Big]\\
&\quad + (3\,n-4)\,A_0(m_2)\Big[C_{2i}\,(p_1,p_2,\{m\}_{345}) - C_{2i}\,(p_1,p_2,0,\{m\}_{45})\Big],
\end{aligned}
\tag{342}
$$

for $i < 4$ and

$$
\begin{aligned}
4\,(n-1)\,m_3^2\,V^{I}_{114} &= -m_3^2\,V^{I}(0|\mu,\mu\,;\,p_1,P,\{m\}_{12345}) - m_{12}^4\,V^{I}_{0}(p_1,P,\{m\}_{12},0,\{m\}_{45})\\
&\quad + m_3^2\,(2\,n\,m_{12}^2 + n\,m_3^2 - 4\,m_1^2)\,V^{I}_{224}(p_1,P,\{m\}_{12345})\\
&\quad - m_3^2\Big[(n-4)\,m_1^2 - 2\,n\,m_2^2\Big]\,V^{I}_{224}(p_1,P,\{m\}_{12},0,\{m\}_{45})\\
&\quad + n\,m_3^2\,m_{12}^4\Big[V^{M}_{224}(p_1,P,\{m\}_{12},0,\{m\}_{45},0) - V^{M}_{224}(p_1,P,\{m\}_{12345},0)\Big]\\
&\quad - n\,m_3^2\,A_0(m_1)\Big[C_{24}\,(p_1,p_2,\{m\}_{345}) - C_{24}\,(p_1,p_2,0,\{m\}_{45})\Big]\\
&\quad - n\,m_{12}^2\,A_0([m_1,m_2])\Big[m_3^2\,C_{24}(2,1,1\,;\,p_1,p_2,0,\{m\}_{45})\\
&\quad - C_{24}\,(p_1,p_2,0,\{m\}_{45}) + C_{24}\,(p_1,p_2,\{m\}_{345})\Big]
\end{aligned}
$$

$$+ (3\,n-4)\, A_0(m_2) \Big[C_{24}\,(p_1,p_2,\{m\}_{345}) - C_{24}\,(p_1,p_2,0,\{m\}_{45}) \Big]$$
$$+ \frac{1}{m_3^2} \Big\{ -A_0(m_1) \Big[m_{123}^2\, C_0(p_1,p_2,\{m\}_{345}) + m_{21}^2\, C_0(p_1,p_2,0,\{m\}_{45}) \Big]$$
$$- A_0(m_2) \Big[m_{213}^2\, C_0(p_1,p_2,\{m\}_{345}) + m_{12}^2\, C_0(p_1,p_2,0,\{m\}_{45}) \Big]$$
$$+ \Big[m_{12}^4 - 2\, m_3^2\,(m_1^2+m_2^2) \Big]\, V_0^I(p_1,P,\{m\}_{12345})\,. \tag{343}$$

This expression requires results from the V^M family, presented in Section B.3.

B.3 $V^M(p_1,P,\{m\}_{12345})$ family

$\boxed{\mathbf{V^M_{2i}}}$ Results were derived in Section 9.3.1, in particular the generalized scalar in Eq.(191). Introduce a vector U_2^M of components

$$U^M_{2\,;\,1} = \frac{1}{2} \Big[-V_0^M l_{134} - V_0^I\,(p_1,P,\{m\}_{12345}) + V_0^I\,(0,P,\{m\}_{12335}) \Big]\,,$$
$$U^M_{2\,;\,2} = \frac{1}{2} \Big[V_0^M\,(l_{154}-P^2) + V_0^I\,(0,p_1,\{m\}_{12334}) - V_0^I\,(0,P,\{m\}_{12335}) \Big]\,; \tag{344}$$

we obtain the following result:

$$\begin{pmatrix} V^M_{21} \\ V^M_{22} \end{pmatrix} = G^{-1}\, U_2^M, \qquad V^M(0\,|\,p_1) = U^M_{2\,;\,1}, \qquad V^M(0\,|\,p_2) = U^M_{2\,;\,2}. \tag{345}$$

$\boxed{\mathbf{V^M_{1i}}}$ We obtain

$$V^M_{1i}(p_1,P,\{m\}_{12345}) = \frac{m_{123}^2}{2m_3^2}\, V^M_{2i}(p_1,P,\{m\}_{12345}) + \frac{m_{12}^2}{2m_3^4} \Big[V^I_{2i}(p_1,P,\{m\}_{12345})$$
$$- V^I_{2i}(p_1,P,\{m\}_{12},0,\{m\}_{45}) \Big] - \frac{A_0([m_1,m_2])}{2m_3^4} \Big[C_{1i}(2,1,1,p_1,p_2,m_3,\{m\}_{345})\, m_3^2$$
$$+ C_{1i}(p_1,p_2,\{m\}_{345}) - C_{1i}(p_1,p_2,0,\{m\}_{45}) \Big]. \tag{346}$$

$\boxed{\mathbf{V^M_{22i}}}$ For rank two tensors (see Section 9.3.2) we introduce a vector U^M_{22} with components

$$U^M_{22\,;\,1} = -\frac{1}{2} \Big[l_{134}\, V^M_{22} + V^I_{22}(p_1,P,\{m\}_{12345}) - V^I_{22}(0,P,\{m\}_{12335}) \Big],$$
$$U^M_{22\,;\,2} = -V^M_{224} + \frac{1}{2}\,(p_1^2 - l_{P45})\, V^M_{22} - \frac{1}{2}\, V^I_{22}(0,P,\{m\}_{12335})\,; \tag{347}$$

we obtain the following result:

$$\begin{pmatrix} V^M_{223} \\ V^M_{222} \end{pmatrix} = G^{-1}\, U^M_{22}. \tag{348}$$

Introduce a vector W^M_{22} with components

$$W^M_{22\,;\,1} = -V^M_{224} - \frac{1}{2} \Big[V^I_{21}(p_1,P,\{m\}_{12345}) - V^I_{22}(0,P,\{m\}_{12335}) + l_{134}\, V^M_{21} \Big],$$
$$W^M_{22\,;\,2} = -\frac{1}{2} \Big[-(p_1^2 - l_{P45})\, V^M_{21} + V^I_{22}(0,P,\{m\}_{12335}) - V^I_{22}(0,p_1,\{m\}_{12334}) \Big]\,; \tag{349}$$

we obtain the following result:

$$\begin{pmatrix} V^M_{221} \\ V^M_{223} \end{pmatrix} = G^{-1}\, W^M_{22}. \tag{350}$$

Furthermore we derive

$$(2-n)\, V^{M}_{224} = V^{M}_{0}\, m_3^2 - \frac{1}{2}\, V^{M}_{21}\, l_{134} + \frac{1}{2}\, V^{M}_{22}\, (p_1^2 - l_{P45}) - V^{I}_{0}(p_1, P, \{m\}_{12345}) - \frac{1}{2}\, V^{I}_{21}(p_1, P, \{m\}_{12345}) \tag{351}$$

For tensor integrals with saturated indices we obtain

$$\begin{aligned}
V^{M}(0\,|\,\mu,\mu) &= -V^{M}_{0}\, m_3^2 + V^{I}_{0}(p_1, P, \{m\}_{12345}),\\
p_2^2\, V^{M}(0\,|\,p_1,p_1) &= \frac{1}{2}\Big[-l_{134}\,(V^{M}_{21}\, D_1 + V^{M}_{22}\, D_2) - V^{I}_{21}(p_1, P, \{m\}_{12345})\, D_1\\
&\quad - V^{I}_{22}(p_1, P, \{m\}_{12345})\, D_2 + V^{I}_{22}(0, P, \{m\}_{12335})\,(D_1 + D_2)\Big],\\
p_1^2\, V^{M}(0\,|\,p_2,p_2) &= \frac{1}{2}\Big[-V^{M}_{21}\, l_{134}\, p_{12}^2 + V^{M}_{22}\,(D\, p_1^2 - D\, l_{P45} - D_2\, l_{134}) - V^{I}_{21}(p_1, P, \{m\}_{12345})\, p_{12}^2\\
&\quad - V^{I}_{22}(p_1, P, \{m\}_{12345})\, D_2 - V^{I}_{22}(0, P, \{m\}_{12335})\,(2\, D - D_1 - D_2)\Big],\\
V^{M}(0\,|\,p_1,p_2) &= \frac{1}{2}\Big[-l_{134}\,(V^{M}_{21}\, p_{12} + V^{M}_{22}\, p_2^2) - V^{I}_{21}(p_1, P, \{m\}_{12345})\, p_{12} - V^{I}_{22}(p_1, P, \{m\}_{12345})\, p_2^2\\
&\quad + V^{I}_{22}(0, P, \{m\}_{12335})\, p_2\cdot P\Big].
\end{aligned} \tag{352}$$

$\boxed{\mathbf{V^{M}_{12i}}}$ For tensor integrals in the 12 group we obtain

$$\begin{aligned}
V^{M}_{12i} &= V^{M}_{22i}(p_1, P, \{m\}_{12345})\, \frac{m_{312}^2}{2m_3^2} + \frac{m_{12}^2}{2\, m_3^4}\Big[V^{I}_{22i}(p_1, P, \{m\}_{12345})\\
&\quad - V^{I}_{22i}(p_1, P, \{m\}_{12}, 0, \{m\}_{45})\Big] + \Delta\, V^{M}_{12i}(p_1, P, \{m\}_{12345}),
\end{aligned} \tag{353}$$

$$\begin{aligned}
\Delta\, V^{M}_{12i} &= -\frac{A_0([m_1,m_2])}{2m_3^4}\Big[C_{2i}(p_1, p_2, \{m\}_{345}) - C_{2i}(p_1, p_2, 0, \{m\}_{45})\\
&\quad + m_3^2\, C_{2i}(2,1,1\,;\, p_1, p_2, \{m\}_{345})\Big], \qquad i = 1\cdots 4.
\end{aligned} \tag{354}$$

$\boxed{\mathbf{V^{M}_{11i}}}$ For tensor integrals in the 11 group we obtain

$$\begin{aligned}
V^{M}_{11i} &= \frac{1}{4\,(n-1)}\,\frac{1}{m_3^4}\Big\{(n\, m_{123}^4 - 4\, m_1^2\, m_3^2)\, V^{M}_{22i}(p_1, P, \{m\}_{12345}) + V^{M}_{22i}(p_1, P, \{m\}_{12}, 0, \{m\}_{45})\, n\, m_{12}^4\\
&\quad + \frac{2}{m_3^2}\,(2\, m_1^2\, m_3^2 - n\, m_{12}^2\, m_{123}^2)\Big[V^{I}_{22i}(p_1, P, \{m\}_{12}, 0, \{m\}_{45}) - V^{I}_{22i}(p_1, P, \{m\}_{12345})\Big] + \Delta V^{M}_{11i}\Big\}.
\end{aligned} \tag{355}$$

For $i < 4$ we have

$$\begin{aligned}
\Delta V^{M}_{11i} &= n\,\Big(2\,\frac{m_{123}^2}{m_3^2} - 1\Big)\, A_0([m_1,m_2])\Big[C_{2i}(p_1, p_2, \{m\}_{345}) - C_{2i}(p_1, p_2, 0, \{m\}_{45})\Big]\\
&\quad - n\, A_0([m_1,m_2])\Big[m_{123}^2\, C_{2i}(2,1,1\,;\, p_1, p_2, \{m\}_{345}) + m_{12}^2\, C_{2i}(2,1,1\,;\, p_1, p_2, 0, \{m\}_{45})\Big]\\
&\quad + 2\,(n-2)\, A_0(m_2)\Big[C_{2i}(p_1, p_2, \{m\}_{345}) - C_{2i}(p_1, p_2, 0, \{m\}_{45}) + m_3^2\, C_{2i}(2,1,1\,;\, p_1, p_2, \{m\}_{345})\Big]
\end{aligned} \tag{356}$$

$$\begin{aligned}
\Delta V^{M}_{114} &= n\,\Big(2\,\frac{m_{123}^2}{m_3^2} - 1\Big)\, A_0([m_1,m_2])\Big[C_{24}(p_1, p_2, \{m\}_{345}) - C_{24}(p_1, p_2, 0, \{m\}_{45})\Big]\\
&\quad - n\, A_0([m_1,m_2])\Big[m_{123}^2\, C_{24}(2,1,1\,;\, p_1, p_2, \{m\}_{345}) + m_{12}^2\, C_{24}(2,1,1\,;\, p_1, p_2, 0, \{m\}_{45})\Big]\\
&\quad + 2\,(n-2)\, A_0(m_2)\Big[C_{24}(p_1, p_2, \{m\}_{345}) - C_{24}(p_1, p_2, 0, \{m\}_{45}) + m_3^2\, C_{24}(2,1,1\,;\, p_1, p_2, \{m\}_{345})\Big]\\
&\quad - m_{12}^2\, A_0([m_1,m_2])\Big[C_0(p_1, p_2, \{m\}_{345}) - C_0(p_1, p_2, 0, \{m\}_{45})\Big]\\
&\quad - m_3^2\, C_0(2,1,1\,;\, p_1, p_2, \{m\}_{345})\Big[m_{123}^2\, A_0(m_1) + m_{213}^2\, A_0(m_2)\Big].
\end{aligned} \tag{357}$$

B.4 $V^G(p_1, p_1, P, \{m\}_{12345})$ family

$\boxed{\mathbf{V^G_{i1}}}$ Results were derived in Section 9.4.1, in particular the generalized scalar in Eq.(199). Introduce a vector U^G_1 with components

$$
\begin{aligned}
U^G_{1\,;\,1} &= \frac{1}{2}\Big[-l_{112}\, V^G_0 + V^E_0(p_1, P, \{m\}_{1345}) - V^E_0(0, p_2, \{m\}_{2345})\Big], \\
U^G_{1\,;\,2} &= \frac{1}{2}\Big[(-l_{245} - 2p_{12})V^G_0 - V^E_0(-p_2, -P, \{m\}_{5321}) + V^E_0(0, -p_1, \{m\}_{4321})\Big].
\end{aligned} \tag{358}
$$

Referring to Eq.(199) we obtain

$$
\begin{aligned}
V^G_{11} &= \frac{1}{p_1^2}\,\Big[U^G_{1\,;\,1} + \omega^2\, p_{12}\, V_G^{1,1|1,2|2}\Big|_{n=6-\epsilon}\Big], \\
V^G_{21} &= \frac{1}{p_{12}}\,\Big\{U^G_{1\,;\,2} + \,\omega^2\, p_2^2\,\Big[V_G^{1,2|1,2|1} + V_G^{2,1|1,2|1} + \,V_G^{1,1|1,2|2}\Big]\Big|_{n=6-\epsilon}\Big\}.
\end{aligned} \tag{359}
$$

$\boxed{\mathbf{V^G_{i2}}}$ Furthermore we find

$$
V^G_{12} = -\omega^2\, V_G^{1,1|1,2|2}\Big|_{n=6-\epsilon}, \quad V^G_{22} = -\,\omega^2\,\Big[V_G^{1,2|1,2|1} + V_G^{2,1|1,2|1} + \,V_G^{1,1|1,2|2}\Big]\Big|_{n=6-\epsilon}. \tag{360}
$$

$\boxed{\mathbf{V^G_{22i}}}$ For rank two tensors (see Section 9.4.2) we introduce a vector U^G_{22} with components

$$
\begin{aligned}
U^G_{22\,;\,1} &= V^G_{224}\,(1-n) - V^G_0\,(p_1^2 + m_4^2) - 2\,V^G_{21}\,p_1^2 + \frac{1}{2}\,\Big[V^G_{22}\,(P^2 - 4\,p_{12} - l_{154}) \\
&\quad + 2\,V^E_0(-p_2, -P, \{m\}_{5321}) - V^E_0(0, -p_1, \{m\}_{4321}) - V^E_{12}(-p_2, -P, \{m\}_{5321})\Big], \\
U^G_{22\,;\,2} &= \frac{1}{2}\,\Big[-V^G_{21}\,(P^2 - l_{154}) + V^E_0(-p_2, -P, \{m\}_{5321}) \\
&\quad - V^E_0(0, -p_1, \{m\}_{4321}) + V^E_{11}(-p_2, -P, \{m\}_{5321}) - V^E_{11}(0, -p_1, \{m\}_{4321})\Big]\,;
\end{aligned} \tag{361}
$$

we obtain the following result:

$$
\begin{pmatrix} V^G_{221} \\ V^G_{223} \end{pmatrix} = G^{-1}\, U^G_{22}, \tag{362}
$$

$$
\begin{aligned}
p_2^2\, V^G_{222} &= -V^G_{224} - V^G_{223}\, p_{12} - \frac{1}{2}\,\Big[V^G_{22}\,(P^2 - l_{154}) - V^E_0(0, -p_1, \{m\}_{4321}) - V^E_{12}(-p_2, -P, \{m\}_{5321})\Big], \\
V^G_{224} &= \frac{1}{2}\,\omega^2\, V_G^{1,1|1,1|2}\Big|_{n=6-\epsilon}.
\end{aligned} \tag{363}
$$

When indices are saturated we obtain

$$
\begin{aligned}
V^G(0\,|\,\mu,\mu) &= -V^G_0\,(p_1^2 + m_4^2) - 2\,V^G_{21}\,p_1^2 - 2\,V^G_{22}\,p_{12} + V^E_0(-p_2, -P, \{m\}_{5321}), \\
V^G(0\,|\,p_1,p_1) &= -V^G_0\,(p_1^2 + m_4^2)\,p_1^2 - V^G_{224}\,(n-1)\,p_1^2 \\
&\quad - 2\,V^G_{21}\,p_1^4 - 2\,V^G_{22}\,D_3 + V^E_0(-p_2, -P, \{m\}_{5321})\,p_1^2, \\
p_1^2\, V^G(0\,|\,p_2,p_2) &= -V^G_0\,(p_1^2 + m_4^2)\,p_{12}^2 - V^G_{224}\,\Big[D + (n-1)\,p_{12}^2\Big] \\
&\quad - 2\,V^G_{21}\,D_3\,p_{12} - V^G_{22}\,\Big[D\,(P^2 - l_{154}) + 2\,p_{12}^3\Big] + V^E_0(-p_2, -P, \{m\}_{5321})\,p_{12}^2 \\
&\quad + V^E_0(0, -p_1, \{m\}_{4321})\,D + V^E_{12}(-p_2, -P, \{m\}_{5321})\,D, \\
p_{12}\, V^G(0\,|\,p_1,p_2) &= -V^G_0\,(p_1^2 + m_4^2)\,p_{12}^2 - V^G_{224}\,\Big[D + (n-1)\,p_{12}^2\Big] \\
&\quad - 2\,V^G_{21}\,D_3\,p_{12} - \frac{1}{2}\,V^G_{22}\,\Big[D\,(P^2 - l_{154}) + 4\,p_{12}^3\Big] + V^E_0(-p_2, -P, \{m\}_{5321})\,p_{12}^2 \\
&\quad + \frac{1}{2}\,V^E_0(0, -p_1, \{m\}_{4321})\,D + \frac{1}{2}\,V^E_{12}(-p_2, -P, \{m\}_{5321})\,D.
\end{aligned} \tag{364}
$$

$\boxed{\mathbf{V^G_{12i}}}$ Introduce a vector U^{G}_{12} with components

$$
\begin{aligned}
U^{G}_{12\,;\,1} &= \frac{1}{2}\Big[-V^{G}_{22}\,l_{112} + V^{E}_{21}(p_1,P,\{m\}_{1345}) - V^{E}_{21}(0,p_2,\{m\}_{2345})\Big],\\
U^{G}_{12\,;\,2} &= \frac{1}{2}\Big[V^{G}(0\,|\,\mu\mu) + V^{G}_{0}\,m^2_{31} + V^{G}_{21}\,l_{112} + 2\,V^{G}_{124}\,(n-1)\\
&\quad + 2\,V^{E}_{0}(0,p_2,\{m\}_{2345}) + B_0(p_1,\{m\}_{12})\,B_0(p_2,\{m\}_{45}) - V^{E}_{22}(p_1,P,\{m\}_{1345})\Big]\,;
\end{aligned}
\tag{365}
$$

we obtain the following result:

$$
\begin{pmatrix} V^{G}_{123} \\ V^{G}_{122} \end{pmatrix} = G^{-1}\,U^{G}_{12}. \tag{366}
$$

Introduce a vecror W^{G}_{12} with components

$$
\begin{aligned}
W^{G}_{12\,;\,1} &= \frac{1}{2}\Big[2\,V^{G}_{124}\,(n-1) - V^{G}_{12}\,(-P^2+l_{145}) + V^{G}_{0}\,m^2_{31} + V^{G}(0\,|\,\mu\mu) - V^{E}_{22}(-p_2,-P,\{m\}_{5321})\\
&\quad - V^{E}_{0}(-p_2,-P,\{m\}_{5321}) + V^{E}_{0}(0,p_2,\{m\}_{2345}) + B_0(p_1,\{m\}_{12})\,B_0(p_2,\{m\}_{45})\Big],\\
W^{G}_{12\,;\,1} &= \frac{1}{2}\Big[-V^{G}_{11}\,(P^2-l_{145}) + V^{E}_{0}(-p_2,-P,\{m\}_{5321}) - V^{E}_{0}(0,-p_1,\{m\}_{4321})\\
&\quad - V^{E}_{11}(0,-p_1,\{m\}_{4321}) + V^{E}_{12}(-p_2,-P,\{m\}_{5321})\Big]\,;
\end{aligned}
\tag{367}
$$

we obtain the following result;

$$
\begin{pmatrix} V^{G}_{121} \\ V^{G}_{123} \end{pmatrix} = G^{-1}\,W^{G}_{12}. \tag{368}
$$

Furthermore we get

$$
\begin{aligned}
V^{G}_{125} &= -\frac{1}{p_{12}}\Big\{V^{G}_{122}\,p^2_2 + V^{G}_{124} + \frac{1}{2}\Big[-V^{E}_{0}(-p_2,-P,\{m\}_{5321}) - V^{E}_{22}(-p_2,-P,\{m\}_{5321}) + V^{G}_{12}\,(P^2-l_{145})\Big]\Big\},\\
V^{G}_{124} &= \frac{1}{2}\,\omega^2\,V^{1,1|1,1|2}_{G}\Big|_{n=6-\epsilon}.
\end{aligned}
\tag{369}
$$

For saturated indices we have

$$
\begin{aligned}
V^{G}(\mu\,|\,\mu) &= \frac{1}{2}\Big[V^{G}_{0}\,m^2_{31} + V^{G}(0\,|\,\mu,\mu) + V^{E}_{0}(0,p_2,\{m\}_{2345}) + B_0(p_1,\{m\}_{12})\,B_0(p_2,\{m\}_{45})\Big],\\
V^{G}(p_1\,|\,p_1) &= \frac{1}{2}\Big[-l_{112}\,(V^{G}_{22}\,p_{12} + V^{G}_{21}\,p^2_1) + V^{E}_{21}(p_1,P,\{m\}_{1345})\,p_{12} - V^{E}_{21}(0,p_2,\{m\}_{2345})\,p_{12}\\
&\quad - V^{E}_{0}(0,p_2,\{m\}_{2345})\,p^2_1 + V^{E}_{22}(p_1,P,\{m\}_{1345})\,p^2_1\Big],\\
p^2_1\,V^{G}(p_2\,|\,p_2) &= \frac{1}{2}\Big[-V^{G}_{22}\,D_2\,l_{112} + V^{G}_{12}\,D\,(-P^2+l_{145}) + V^{G}_{21}\,(D-D_1)\,l_{112}\\
&\quad + V^{E}_{21}(p_1,P,\{m\}_{1345})\,D_2 - V^{E}_{21}(0,p_2,\{m\}_{2345})\,D_2 + V^{E}_{0}(-p_2,-P,\{m\}_{5321})\,D\\
&\quad - V^{E}_{0}(0,p_2,\{m\}_{2345})\,p^2_{12} + V^{E}_{22}(p_1,P,\{m\}_{1345})\,p^2_{12} + V^{E}_{22}(-p_2,-P,\{m\}_{5321})\,D\Big],\\
V^{G}(p_1\,|\,p_2) &= \frac{1}{2}\Big[-l_{112}\,(V^{G}_{22}\,p^2_2 + V^{G}_{21}\,p_{12}) + V^{E}_{21}(p_1,P,\{m\}_{1345})\,p^2_2 - V^{E}_{21}(0,p_2,\{m\}_{2345})\,p^2_2\\
&\quad - V^{E}_{0}(0,p_2,\{m\}_{2345})\,p_{12} + V^{E}_{22}(p_1,P,\{m\}_{1345})\,p_{12}\Big],\\
p_{12}\,V^{G}(p_2\,|\,p_1) &= \frac{1}{2}\Big[-l_{112}\,(V^{G}_{22}\,D_2 + V^{G}_{21}\,D_1) + 2\,V^{G}_{124}\,D\,(n-2) - V^{G}_{0}\,D\,m^2_{31} - V^{G}(0\,|\,\mu\mu)\,D\\
&\quad + V^{G}_{12}\,D\,(-P^2+l_{145}) + V^{E}_{21}(p_1,P,\{m\}_{1345})\,D_2 - V^{E}_{21}(0,p_2,\{m\}_{2345})\,D_2\\
&\quad + V^{E}_{0}(-p_2,-P,\{m\}_{5321})\,D - V^{E}_{0}(0,p_2,\{m\}_{2345})\,(D+D_1) + V^{E}_{22}(p_1,P,\{m\}_{1345})\,D_1\\
&\quad + V^{E}_{22}(-p_2,-P,\{m\}_{5321})\,D - B_0(p_1,\{m\}_{12})\,B_0(p_2,\{m\}_{45})\,D\Big].
\end{aligned}
\tag{370}
$$

$\boxed{\mathbf{V^{G}_{11i}}}$ For form factors belonging to the 11 group we have

$$
\begin{aligned}
(n-1)\,p_1^4\,V^{G}_{111} &= \Big\{V^{G}_{112}\,\Big[(n-1)\,D_1-(n-2)\,D\Big]+V^{G}_{0}\,p_1^2\,m_1^2+\frac{1}{2}\,\Big[\\
&\quad -V^{G}_{11}\,n\,p_1^2\,l_{112}+V^{E}_{12}(p_1,P,\{m\}_{1345})\,n\,p_1^2+(n-2)\,\Big[V^{G}_{12}\,p_{12}\,l_{112}+V^{E}_{0}(0,p_2,\{m\}_{2345})\,p_1^2\\
&\quad -V^{E}_{11}(p_1,P,\{m\}_{1345})\,p_{12}+V^{E}_{11}(0,p_2,\{m\}_{2345})\,p_{12}\Big]\Big\},\\
p_1^2\,V^{G}_{113} &= \frac{1}{2}\,\Big[-2\,V^{G}_{112}\,p_{12}-V^{G}_{12}\,l_{112}+V^{E}_{11}(p_1,P,\{m\}_{1345})-V^{E}_{11}(0,p_2,\{m\}_{2345})\Big],\\
(n-1)\,p_1^2\,V^{G}_{114} &= \frac{1}{2}\,\Big[l_{112}\,(V^{G}_{11}\,p_1^2+V^{G}_{12}\,p_{12})-2\,V^{G}_{112}\,D-2\,V^{G}_{0}\,p_1^2\,m_1^2+V^{E}_{0}(0,p_2,\{m\}_{2345})\,p_1^2\\
&\quad -V^{E}_{12}(p_1,P,\{m\}_{1345})\,p_1^2-V^{E}_{11}(p_1,P,\{m\}_{1345})\,p_{12}+V^{E}_{11}(0,p_2,\{m\}_{2345})\,p_{12}\Big],\\
V^{G}_{112} &= 4\,\omega^4\,V_{G}^{1,1|1,3|3}(n=8-\epsilon).
\end{aligned}
\tag{371}
$$

For saturated indices we have

$$
\begin{aligned}
V^{G}(\mu,\mu\,|\,0) &= -V^{G}_{0}\,m_1^2+V^{E}_{0}(0,p_2,\{m\}_{2345}),\\
V^{G}(p_1,p_1\,|\,0) &= \frac{1}{2}\,\Big[-l_{112}\,(V^{G}_{11}\,p_1^2+V^{G}_{12}\,p_{12})+V^{E}_{0}(0,p_2,\{m\}_{2345})\,p_1^2\\
&\quad +V^{E}_{12}(p_1,P,\{m\}_{1345})\,p_1^2+V^{E}_{11}(p_1,P,\{m\}_{1345})\,p_{12}-V^{E}_{11}(0,p_2,\{m\}_{2345})\,p_{12}\Big],\\
(n-1)\,p_1^4\,V^{G}(p_2,p_2\,|\,0) &= \frac{1}{2}\,\Big\{2\,V^{G}_{112}\,(n-2)\,D^2-2\,V^{G}_{0}\,D\,p_1^2\,m_1^2+V^{G}_{11}\,p_1^2\,l_{112}\,\Big[D-(n-1)\,p_{12}^2\Big]\\
&\quad +V^{G}_{12}\,p_{12}\,l_{112}\,\Big[D-(n-1)\,(D+D_3)\Big]+V^{E}_{0}(0,p_2,\{m\}_{2345})\,p_1^2\,\Big[D+(n-1)\,p_{12}^2\Big]\\
&\quad +(n-1)\,p_{12}\,(D+D_1)\,\Big[V^{E}_{11}(p_1,P,\{m\}_{1345})\,-V^{E}_{11}(0,p_2,\{m\}_{2345})\Big]\\
&\quad -V^{E}_{12}(p_1,P,\{m\}_{1345})\,p_1^2\,\Big[D-(n-1)\,p_{12}^2\Big]-D\,p_{12}\,\Big[V^{E}_{11}(p_1,P,\{m\}_{1345})\\
&\quad -V^{E}_{11}(0,p_2,\{m\}_{2345})\Big]\Big\},\\
V^{G}(p_1,p_2\,|\,0) &= \frac{1}{2}\,\Big\{-l_{112}\,(V^{G}_{11}\,p_{12}+V^{G}_{12}\,p_2^2)+p_{12}\,\Big[V^{E}_{0}(0,p_2,\{m\}_{2345})\\
&\quad +V^{E}_{12}(p_1,P,\{m\}_{1345})\Big]+p_2^2\,\Big[V^{E}_{11}(p_1,P,\{m\}_{1345})-V^{E}_{11}(0,p_2,\{m\}_{2345})\Big]\Big\}.
\end{aligned}
\tag{372}
$$

B.5 $V^{K}(P,p_1,P,\{m\}_{123456})$ family

$\boxed{\mathbf{V^{K}_{1i}}}$ Results were derived in Section 9.5.1. Referring to Eq.(226) the vector integrals are

$$
V^{K}_{11} = \frac{p_2\cdot P}{P^2}\,\omega^2\,V_{K}^{1,1|1,2,1|2}\Big|_{n=6-\epsilon},\qquad V^{K}_{12} = V^{K}_{11}+\omega^2\,V_{K}^{1,1|1,2,1|2}\Big|_{n=6-\epsilon}.
\tag{373}
$$

$\boxed{\mathbf{V^{K}_{2i}}}$ Introduce a vector U^{K}_{2} with components

$$
\begin{aligned}
U^{K}_{2\,;\,1} &= \frac{1}{2}\Big[l_{145}V^{K}_{0}+V^{G}_{0}(P,P,p_1,\{m\}_{12365})-V^{G}_{0}(P,P,0,\{m\}_{12364})\Big],\\
U^{K}_{2\,;\,2} &= -\frac{1}{2}\Big[l_{P46}V^{K}_{0}+V^{G}_{0}(P,P,p_1,\{m\}_{12365})-V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big]\,;
\end{aligned}
\tag{374}
$$

we obtain the following result:

$$
\begin{pmatrix} V^{K}_{21}\\ V^{K}_{22}\end{pmatrix} = G^{-1}\,W^{K}_{2},\quad W^{K}_{2\,;\,1} = U^{K}_{2\,;\,1},\quad W^{K}_{2\,;\,2} = U^{K}_{2\,;\,2}-U^{K}_{2\,;\,1}.
\tag{375}
$$

$\boxed{\mathbf{V^K_{22i}}}$ For rank two tensors (see Section 9.5.2) we introduce a vector U^{K}_{22} with components

$$
\begin{aligned}
U^{K}_{22\,;\,1} &= -\frac{1}{2}\Big[V^{K}_{22}\,l_{145} - V^{G}_{21}(P,P,0,\{m\}_{12364}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\Big],\\
U^{K}_{22\,;\,2} &= -\frac{1}{2}\Big[2\,V^{K}_{224} + V^{K}_{22}\,(P^2 - l_{165}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{21}(P,P,0,\{m\}_{12364}) - V^{G}_{22}(P,P,0,\{m\}_{12364})\Big]\,;
\end{aligned}
\tag{376}
$$

we obtain the following result:

$$
\begin{pmatrix} V^{K}_{223} \\ V^{K}_{222} \end{pmatrix} = G^{-1}\,U^{K}_{22}.
\tag{377}
$$

Introduce a vector W^{K}_{22} with components

$$
\begin{aligned}
W^{K}_{22\,;\,1} &= -\frac{1}{2}\Big[2\,V^{K}_{224} + V^{K}_{21}\,l_{145}\\
&\quad + V^{G}_{21}(P,P,p_1,\{m\}_{12365}) - V^{G}_{21}(P,P,0,\{m\}_{12364}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\Big],\\
W^{K}_{22\,;\,2} &= -\frac{1}{2}\Big[V^{K}_{21}\,(P^2 - l_{165}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{23145}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{21}(P,P,0,\{m\}_{12364}) - V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{22}(P,P,0,\{m\}_{12364})\Big]\,;
\end{aligned}
\tag{378}
$$

we obtain

$$
\begin{pmatrix} V^{K}_{221} \\ V^{K}_{223} \end{pmatrix} = G^{-1}\,W^{K}_{22},
\tag{379}
$$

and also

$$
\begin{aligned}
V^{K}_{224} &= \frac{1}{2\,(2-n)}\Big[2\,V^{K}_{0}\,m_4^2 - V^{K}_{21}\,l_{145} + V^{K}_{22}\,(-P^2 + l_{165}) - V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - 2\,V^{G}_{0}(P,P,p_1,\{m\}_{12365}) - V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{21}(P,P,p_1,\{m\}_{12365})\Big].
\end{aligned}
\tag{380}
$$

For saturated indices we get

$$
\begin{aligned}
V^{K}(0\,|\,\mu,\mu) &= -V^{K}_{0}\,m_4^2 + V^{G}_{0}(P,P,p_1,\{m\}_{12365}),\\
V^{K}(0\,|\,p_1,p_1) &= \frac{1}{2}\Big\{-l_{145}\,(V^{K}_{21}\,p_1^2 + V^{K}_{22}\,p_{12}) - V^{G}_{21}(P,P,p_1,\{m\}_{12365})\,p_1^2\\
&\quad + p_1\cdot P\,\Big[V^{G}_{21}(P,P,0,\{m\}_{12364}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\Big]\Big\},\\
p_1^2\,V^{K}(0\,|\,p_2,p_2) &= \frac{1}{2}\Big[-V^{K}_{21}\,l_{145}\,p_{12}^2 + V^{K}_{22}\,\Big[(l_{165} - P^2)\,D - D_2\,l_{145}\Big] - V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345})\,D\\
&\quad - V^{G}_{21}(P,P,p_1,\{m\}_{12365})\,p_{12}^2 - V^{G}_{21}(P,P,0,\{m\}_{12364})\,(2\,D - D_1 - D_2)\\
&\quad + V^{G}_{22}(P,P,0,\{m\}_{12364})\,(2\,D - D_1 - D_2) - V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\,D\Big],\\
V^{K}(0\,|\,p_1,p_2) &= \frac{1}{2}\Big\{-l_{145}\,(V^{K}_{21}\,p_{12} + V^{K}_{22}\,p_2^2) - V^{G}_{21}(P,P,p_1,\{m\}_{12365})\,p_{12}\\
&\quad + p_2\cdot P\,\Big[V^{G}_{21}(P,P,0,\{m\}_{12364}) + V^{G}_{22}(P,P,0,\{m\}_{12364})\Big]\Big\}.
\end{aligned}
\tag{381}
$$

$\boxed{\mathbf{V^K_{12i}}}$ Introduce a vector U^{K}_{12} with components

$$
\begin{aligned}
U^{K}_{12\,;\,1} &= -\frac{1}{2}\Big[V^{K}_{12}\,l_{145} - V^{G}_{12}(P,P,p_1,\{m\}_{12365})\\
&\quad + V^{G}_{12}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{G}_{11}(P,P,0,\{m\}_{12364})\Big],
\end{aligned}
$$

$$
\begin{aligned}
U^{K}_{12\,;\,2} = &-\frac{1}{2}\Big[2\,V^{K}_{124} + V^{K}_{12}\,(P^2 - l_{165}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{12}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{11}(P,P,0,\{m\}_{12364})\Big]\,;
\end{aligned}
\tag{382}
$$

we obtain the following result:

$$
\begin{pmatrix} V^{K}_{125} \\ V^{K}_{122} \end{pmatrix} = G^{-1}\,U^{K}_{12}.
\tag{383}
$$

Introduce a vector W^{K}_{12} with components

$$
\begin{aligned}
W^{K}_{12\,;\,1} = &-\frac{1}{2}\Big[2\,V^{K}_{124} + V^{K}_{11}\,l_{145} + V^{G}_{11}(P,P,p_1,\{m\}_{12365})\Big], \\
W^{K}_{12\,;\,2} = &-\frac{1}{2}\Big[V^{K}_{11}\,(P^2 - l_{165}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{12}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{11}(P,P,0,\{m\}_{12364})\Big]\,;
\end{aligned}
\tag{384}
$$

we obtain

$$
\begin{pmatrix} V^{K}_{121} \\ V^{K}_{123} \end{pmatrix} = G^{-1}\,W^{K}_{12},
\tag{385}
$$

and also

$$
\begin{aligned}
(2-n)\,V^{K}_{124} = &\frac{1}{2}\Big[V^{K}_{0}\,m^2_{134} - V^{K}_{12}\,(P^2 - l_{165}) - V^{K}_{11}\,l_{145} - V^{G}_{0}(P,P,p_1,\{m\}_{12365}) \\
&- V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654}) - V^{G}_{11}(P,P,p_1,\{m\}_{12365}) \\
&- V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) - B_0(P,\{m\}_{12})\,C_0(p_1,p_2,\{m\}_{456})\Big].
\end{aligned}
\tag{386}
$$

For saturated indices we obtain

$$
\begin{aligned}
V^{K}(\mu\,|\,\mu) = &\frac{1}{2}\Big[-V^{K}_{0}\,m^2_{134} + V^{G}_{0}(P,P,p_1,\{m\}_{12365}) \\
&+ V^{I}_{0}(-p_2,-P,\{m\}_{23654}) + B_0(P,\{m\}_{12})\,C_0(p_1,p_2,\{m\}_{456})\Big], \\
V^{K}(p_1\,|\,p_1) = &\frac{1}{2}\Big\{-l_{145}\,(V^{K}_{11}\,p_1^2 + V^{K}_{12}\,p_{12}) + p_{12}\,V^{G}_{12}(P,P,p_1,\{m\}_{12365}) \\
&- p_1\cdot P\,\Big[V^{G}_{12}(P,P,0,\{m\}_{12364}) - V^{G}_{11}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(P,P,p_1,\{m\}_{12365})\Big]\Big\}, \\
V^{K}(p_2\,|\,p_2) = &\frac{1}{2}\Big\{-(P^2 - l_{165})\,(V^{K}_{11}\,p_{12} + V^{K}_{12}\,p_2^2) \\
&- p_2\cdot P\,\Big[V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{12}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) \\
&+ V^{G}_{11}(P,P,0,\{m\}_{12364})\Big] + V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345})\,p_{12}\Big\}, \\
V^{K}(p_1\,|\,p_2) = &\frac{1}{2}\Big\{-(P^2 - l_{165})\,(V^{K}_{11}\,p_1^2 + V^{K}_{12}\,p_{12}) \\
&- p_1\cdot P\,\Big[V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{12}(P,P,0,\{m\}_{12364}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) \\
&+ V^{G}_{11}(P,P,0,\{m\}_{12364})\Big] + V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345})\,p_1^2\Big\}, \\
V^{K}(p_2\,|\,p_1) = &\frac{1}{2}\Big\{-l_{145}\,(V^{K}_{11}\,p_{12} + V^{K}_{12}\,p_2^2) - p_2\cdot P\,\Big[V^{G}_{11}(P,P,p_1,\{m\}_{12365}) \\
&+ V^{G}_{12}(P,P,0,\{m\}_{12364}) - V^{G}_{11}(p,p,0,\{m\}_{12364})\Big] + p_2^2\,\Big[V^{G}_{12}(P,P,p_1,\{m\}_{12365})\Big\}
\end{aligned}
\tag{387}
$$

$\boxed{\mathbf{V^{K}_{11i}}}$ for the 11 group we introduce auxiliary quantities (they only appear in the present subsection)

$$
V^{K}_{11A} = 4\,\omega^4\,V^{1,1|1,3,1|3}_{K}\Big|_{n=8-\epsilon},
$$

$$
\begin{aligned}
v_1^K &= -m_1^2\, V_0^K + V_0^I(-P, -p_2, \{m\}_{23645}), \\
v_2^K &= \frac{1}{2}\left[-l_{P12}\, V_{11}^K + V_{11}^I(p_1, P, \{m\}_{13456}) + V_{12}^I(-P, -p_2, \{m\}_{23645}) + V_0^I(-P, -p_2, \{m\}_{23645})\right]. \\
v_3^K &= \frac{1}{2}\left[-l_{P12}\, V_{12}^K + V_{12}^I(p_1, P, \{m\}_{13456}) + V_{11}^I(-P, -p_2, \{m\}_{23645}) + V_0^I(-P, -p_2, \{m\}_{23645})\right],
\end{aligned} \tag{388}
$$

to obtain

$$
\begin{aligned}
V_{111}^K &= 2\, V_{113}^K - V_{112}^K + V_{11A}^K, \quad P^2\, V_{112}^K = V_{113}^K\, P^2 - v_2^K + v_3^K + \frac{1}{2}\, V_{11A}^K\, (P^2 + p_1^2 - p_2^2), \\
(n-1)\, P^2\, V_{113}^K &= \frac{1}{4}\Big\{2\, \frac{v_3^K - v_2^K}{P^2}\, (n-2)\, (p_1^2 - p_2^2) - 4\, v_1^K + 2\, v_2^K\, n + 2\, v_3^K\, n \\
&\quad + V_{11A}^K\left[\frac{n-2}{P^2}\, (p_1^2 - p_2^2)^2 - P^2\, n + 2\, (p_1^2 + p_2^2)\right]\Big\}, \\
V_{114}^K &= -V_{113}^K\, p_1 \cdot P - V_{112}^K\, p_2 \cdot P + v_3^K.
\end{aligned} \tag{389}
$$

B.6 $V^H(-p_2, p_1, -p_2, -p_1, \{m\}_{123456})$ family

$\boxed{\mathbf{V^H_{ij}}}$ Results were derived in Section 9.6.1. Referring to Eq.(250) we have

$$
V_{22}^H = \omega^2\left[V_H^{1,2|1,1|2,1} - V_H^{2,1|1,1|1,2}\right]\Big|_{n=6-\epsilon}, \quad V_{11}^H = \omega^2\left[V_H^{1,1|1,2|1,2} - V_H^{1,1|2,1|2,1}\right]\Big|_{n=6-\epsilon}. \tag{390}
$$

Introduce a vector U^H with components

$$
\begin{aligned}
U_1^H &= \frac{1}{2}\Big[l_{212} V_0^H - V_0^G(p_1, p_1, -p_2, \{m\}_{56134}) + V_0^G(-P, -P, -p_2, \{m\}_{34256})\Big], \\
U_2^H &= \frac{1}{2}\Big[l_{156} V_0^H + V_0^G(p_2, p_2, -p_1, \{m\}_{21634}) - V_0^G(p_2, p_2, -p_1, \{m\}_{12543})\Big],
\end{aligned} \tag{391}
$$

$$
V_{12}^H = \frac{1}{p_2^2}\, (U_1^H - p_{12}\, V_{11}^H), \quad V_{21}^H = \frac{1}{p_1^2}\, (U_2^H - p_{12}\, V_{22}^H). \tag{392}
$$

$\boxed{\mathbf{V^H_{22i}}}$ For rank two tensors (see Section 9.6.2) we introduce a vector U_{22}^H with components

$$
\begin{aligned}
U_{22\,;\,1}^H &= -\frac{1}{2}\Big[-V_{22}^H - V_{11}^G(-p_2, -p_2, p_1, \{m\}_{12543}) + V_{12}^G(-p_2, -p_2, p_1, \{m\}_{12543}) \\
&\quad + V_{21}^G(-p_2, -p_2, p_1, \{m\}_{12543}) - V_{22}^G(-p_2, -p_2, p_1, \{m\}_{12543}) - V_{11}^G(p_2, p_2, -p_1, \{m\}_{21634}) \\
&\quad + V_{12}^G(p_2, p_2, -p_1, \{m\}_{21634}) + V_{21}^G(p_2, p_2, -p_1, \{m\}_{21634}) - V_{22}^G(p_2, p_2, -p_1, \{m\}_{21634})\Big], \\
U_{22\,;\,2}^H &= -\frac{1}{2}\Big[2\, V_0^H + V_{21}^H\, l_{156} + 2\, V_{224}^H\, (n-1) \\
&\quad - V_0^G(p_2, p_2, -p_1, \{m\}_{21634}) - V_{12}^G(p_2, p_2, -p_1, \{m\}_{21634}) + V_{22}^G(p_2, p_2, -p_1, \{m\}_{21634}) \\
&\quad - V_{12}^G(-p_2, -p_2, p_1, \{m\}_{12543}) + V_{22}^G(-p_2, -p_2, p_1, \{m\}_{12543})\Big];
\end{aligned} \tag{393}
$$

we obtain

$$
\begin{pmatrix} V_{223}^H \\ V_{222}^H \end{pmatrix} = G^{-1}\, U_{22}^H, \tag{394}
$$

and also

$$
\begin{aligned}
p_1^2\, V_{221}^H &= -V_{223}^H - V_{224}^H + \frac{1}{2}\Big[V_0^G(p_2, p_2, -p_1, \{m\}_{21634}) + V_{21}^H\, l_{156} - V_{12}^G(p_2, p_2, -p_1, \{m\}_{21634}) \\
&\quad - V_{12}^G(-p_2, -p_2, p_1, \{m\}_{12543}) + V_{22}^G(p_2, p_2, -p_1, \{m\}_{21634}) + V_{22}^G(-p_2, -p_2, p_1, \{m\}_{12543})\Big], \\
V_{224}^H &= \frac{\omega^2}{2}\left[V_H^{1,1|1,1|1,2} + V_H^{1,1|1,1|2,1} + V_H^{2,1|1,1|1,1} + V_H^{1,2|1,1|1,1}\right]\Big|_{n=6-\epsilon}.
\end{aligned} \tag{395}
$$

When indices are saturated we obtain

$$V^{H}(0\,|\,\mu,\mu) = -V^{H}_{0} + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}),$$

$$\begin{aligned}
V^{H}(0\,|\,p_1,p_1) = \frac{1}{2}\Big\{ & l_{156}\,(V^{H}_{21}\,p_1^2 + V^{H}_{22}\,p_{12}) + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634})\,p_1^2 \\
& - p_1\cdot P\,\Big[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{22}(-p_2,-p_2,p_1,\{m\}_{12543})\Big] \\
& + p_{12}\,\Big[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{21}(-p_2,-p_2,p_1,\{m\}_{12543})\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
p_1^2\,V^{H}(0\,|\,p_2,p_2) = \frac{1}{2}\Big\{ & -2\,V^{H}_{0}\,D\,m_5^2 + l_{156}\,\Big[V^{H}_{21}\,(-2\,D + D_1) + V^{H}_{22}\,D_2\Big] \\
& - 2\,(n-2)\,V^{H}_{224}\,D + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634})\,D_1 \\
& + (2\,D - D_1 - D_2)\,\Big[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{22}(-p_2,-p_2,p_1,\{m\}_{12543})\Big] \\
& + D_2\,\Big[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{21}(-p_2,-p_2,p_1,\{m\}_{12543})\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
V^{H}(0\,|\,p_1,p_2) = \frac{1}{2}\Big\{ & l_{156}\,(V^{H}_{21}\,p_{12} + V^{H}_{22}\,p_2^2) + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634})\,p_{12} \\
& + p_2\cdot P\,\Big[V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& - V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{22}(-p_2,-p_2,p_1,\{m\}_{12543})\Big] \\
& + p_2^2\,\Big[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{21}(-p_2,-p_2,p_1,\{m\}_{12543})\Big]\Big\}. \qquad (396)
\end{aligned}$$

$\boxed{\mathbf{V^{H}_{11i}}}$ Introduce a vector U^{H}_{11} with components

$$\begin{aligned}
U^{H}_{11\,;\,1} = -\frac{1}{2}\,\Big[& 2\,V^{H}_{114}\,(n-1) + 2\,V^{H}_{0}\,m_1^2 + V^{H}_{12}\,l_{212} \\
& - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) - V^{G}_{11}(-P,-P,-p_2,\{m\}_{34256}) \\
& + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234})\Big], \\
U^{H}_{11\,;\,2} = -\frac{1}{2}\,\Big[& -V^{H}_{11}\,l_{212} - V^{G}_{11}(p_1,p_1,-p_2,\{m\}_{56234}) + V^{G}_{21}(p_1,p_1,-p_2,\{m\}_{56234}) \\
& - V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234}) + V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Big]\,; \qquad (397)
\end{aligned}$$

we obtain

$$\begin{pmatrix} V^{H}_{111} \\ V^{H}_{113} \end{pmatrix} = G^{-1}\,U^{H}_{11}, \qquad (398)$$

and also

$$p_2^2\,V^{H}_{112} = \frac{1}{2}\,\Big[V^{H}_{12}\,l_{212} - 2\,V^{H}_{113}\,p_{12} - 2\,V^{H}_{114} + V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) - V^{G}_{11}(-P,-P,-p_2,\{m\}_{34256})$$

$$+ V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Big]. \qquad (399)$$

For saturated indices we obtain

$$V^{H}(\mu,\mu\,|\,0) = -V^{H}_{0}\,m_1^2 + V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}),$$

$$\begin{aligned}
p_2^2\,V^{H}(p_1,p_1\,|\,0) = \frac{1}{2}\Big\{ & 2\,(2-n)\,D\,V^{H}_{114} - 2\,V^{H}_{0}\,D\,m_1^2 \\
& + l_{212}\Big[V^{H}_{12}\,(-2\,D + D_1) + V^{H}_{11}\,D_3\Big] + V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256})\,D_1 \\
& + (2\,D - D_1)\Big[V^{G}_{11}(-P,-P,-p_2,\{m\}_{34256}) - V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256})\Big] \\
& + D_3\Big[V^{G}_{11}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{21}(p_1,p_1,-p_2,\{m\}_{56234})\Big] \\
& + (2\,D - D_1 - D_3)\Big[V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234})\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
V^{H}(p_2,p_2\,|\,0) = \frac{1}{2}\Big\{ & l_{212}\,(V^{H}_{12}\,p_2^2 + V^{H}_{11}\,p_{12}) + p_2^2\Big[V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) \\
& - V^{G}_{11}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256})\Big] \\
& + p_{12}\Big[V^{G}_{11}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{21}(p_1,p_1,-p_2,\{m\}_{56234})\Big] \\
& + p_2\cdot P\Big[V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
V^{H}(p_1,p_2,|\,0) = \frac{1}{2}\Big\{ & l_{212}\,(V^{H}_{12}\,p_{12} + V^{H}_{11}\,p_1^2) + p_{12}\Big[V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) \\
& - V^{G}_{11}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256})\Big] \\
& + p_1^2\Big[V^{G}_{11}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{21}(p_1,p_1,-p_2,\{m\}_{56234})\Big] \\
& + p_1\cdot P\Big[V^{G}_{22}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Big]\Big\}. \qquad (400)
\end{aligned}$$

$\boxed{\mathbf{V^{H}_{12i}}}$ Introduce a vector U^{H}_{12} with components

$$\begin{aligned}
U^{H}_{12\,;\,1} &= -\frac{1}{2}\Big[2\,V^{H}_{124} - V^{H}_{11}\,l_{156} + V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
U^{H}_{12\,;\,2} &= -\frac{1}{2}\Big[-V^{H}_{21}\,l_{212} + V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{11}(p_1,p_1,-p_2,\{m\}_{56234}) \\
&\quad - V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256})\Big]; \qquad (401)
\end{aligned}$$

we obtain

$$\begin{pmatrix} V^{H}_{121} \\ V^{H}_{123} \end{pmatrix} = G^{-1}\,U^{H}_{12}, \qquad (402)$$

and also

$$\begin{aligned}
D\,V^{H}_{122} = \frac{1}{2}\Big\{ & -2\,V^{H}_{124}\,p_1^2 - V^{H}_{12}\,p_{12}\,l_{156} + V^{H}_{22}\,p_1^2\,l_{212} \\
& + p_1^2\Big[V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256}) - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Big] \\
& + p_{12}\Big[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543})\Big]\Big\},
\end{aligned}$$

$$V^{H}_{124} = \frac{\omega^2}{2}\Bigl[V_{H}^{1,1|1,1|1,2} + V_{H}^{1,1|1,1|2,1}\Bigr]\Bigr|_{n=6-\epsilon}. \tag{403}$$

For tensors with saturated indices we have

$$\begin{aligned}
V^{H}(\mu\,|\,\mu) = \frac{1}{2}\Bigl[& 2\,V^{H}_{124}\,(n-2) + V^{H}_{11}\,l_{156} + V^{H}_{22}\,l_{212} \\
& - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234}) - V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
& + V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256})\Bigr],
\end{aligned}$$

$$\begin{aligned}
V^{H}(p_1\,|\,p_1) = \frac{1}{2}\Bigl[& l_{156}\,(V^{H}_{11}\,p_1^2 + V^{H}_{12}\,p_{12}) + p_{12}\Bigl[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543})\Bigr] \\
& - p_1\cdot P\Bigl[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543})\Bigr]\Bigr\},
\end{aligned}$$

$$\begin{aligned}
p_1^2\,V^{H}(p_2\,|\,p_2) = \frac{1}{2}\Bigl[& l_{156}\,(V^{H}_{11}\,p_{12}^2 + V^{H}_{12}\,D_2) + V^{H}_{22}\,D\,l_{212} + D\Bigl[V^{G}_{0}(-P,-P,-p_2,\{m\}_{34256}) \\
& + V^{G}_{21}(-P,-P,-p_2,\{m\}_{34256}) - V^{G}_{12}(p_1,p_1,-p_2,\{m\}_{56234})\Bigr] \\
& - (p_{12}^2 + D_2)\Bigl[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543})\Bigr] \\
& + D_2\Bigl[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
& + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634})\Bigr]\Bigr\},
\end{aligned}$$

$$\begin{aligned}
V^{H}(p_1\,|\,p_2) = V^{H}(p_2\,|\,p_1) = \frac{1}{2}\Bigl[& l_{156}\,(V^{H}_{11}\,p_{12} + V^{H}_{12}\,p_2^2) + p_2^2\Bigl[V^{G}_{11}(p_2,p_2,-p_1,\{m\}_{21634}) \\
& + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{11}(-p_2,-p_2,p_1,\{m\}_{12543})\Bigr] \\
& - p_2\cdot P\Bigl[V^{G}_{12}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{12}(-p_2,-p_2,p_1,\{m\}_{12543})\Bigr]\Bigr\}.
\end{aligned} \tag{404}$$

B.7 Further reduction of rank two integrals

In this Section we collect all combinations of vector form factors that can be further reduced without occurrence of inverse powers of Gram determinants. For each combination we list the equation where the r.h.s. can be found.

$p_1^2\,V^{E}_{21} + p_{12}\,V^{E}_{22}$,			Eq.(134),
$p_1^2\,V^{I}_{21} + p_{12}\,V^{I}_{22}$,	$p_{12}\,V^{I}_{21} + p_2^2\,V^{I}_{22}$,		Eq.(160),
$p_1^2\,V^{M}_{21} + p_{12}\,V^{M}_{22}$,	$p_{12}\,V^{M}_{21} + p_2^2\,V^{M}_{22}$,		Eq.(179),
$p_1^2\,V^{G}_{11} + p_{12}\,V^{G}_{12}$,	$p_{12}\,V^{G}_{21} + p_2^2\,V^{G}_{22}$,		Eq.(201),
$p_1\cdot P\,V^{K}_{11} + p_2\cdot P\,V^{K}_{12}$,	$p_1\cdot P\,V^{K}_{21} + p_2\cdot P\,V^{K}_{22}$,	$p_1^2\,V^{K}_{21} + p_{12}\,V^{K}_{22}$,	Eq.(228),
$p_{12}\,V^{H}_{11} + p_2^2\,V^{H}_{12}$,	$p_1^2\,V^{H}_{21} + p_{12}\,V^{H}_{22}$,		Eq.(254).

Once again we stress that form factors are introduced with respect to a certain basis of vectors which is fully specified by the corresponding list of arguments; therefore, form factors appearing in the reduction of other form factors should be interpreted in the appropriate way. We have collected in Tab. 2 the bases for expansion of tensor integrals occurring in the reduction procedure. A typical example is

$$\begin{aligned}
V^{G}(\mu\,|\,0;p_1,p_1,-p_2,\cdots) &= V^{G}_{11}(p_1,p_1,-p_2,\cdots)\,p_{1\mu} - V^{G}_{12}(p_1,p_1,-p_2,\cdots)\,P_\mu, \\
V^{G}(\mu\,|\,0;-P,-P,-p_2,\cdots) &= -V^{G}_{11}(-P,-P,-p_2,\cdots)\,P_\mu + V^{G}_{12}(-P,-P,-p_2,\cdots)\,p_{1\mu}.
\end{aligned} \tag{405}$$

When reducing recursively all symbols must be interpreted as referred to the appropriated basis, i.e.

Family	argument	$p_{\rm first}$	$p_{\rm second}$
E	$p_2\,,\,P$	p_1	p_2
E	$p_1\,,\,P$	p_2	p_1
E	$0\,,P$	P	0
E	$0\,,\,p_1$	p_1	0
E	$0\,,\,p_2$	p_2	0
E	$-p_2\,,\,-P$	$-p_1$	$-p_2$
E	$0\,,\,-p_1$	$-p_1$	0
I	$p_1\,,\,P$	p_1	p_2
I	$p_2\,,\,P$	p_2	p_1
I	$0\,,\,p_1$	0	p_1
I	$0\,,\,P$	0	P
I	$-p_2\,,\,-P$	$-p_2$	$-p_1$
I	$-P\,,\,-p_2$	$-p_2$	$-p_1$
I	$p_1\,,\,0$	p_1	$-p_1$
G	$p_1\,,\,p_1\,,\,P$	p_1	p_2
G	$P\,,\,P\,,\,p_1$	P	$-p_2$
G	$-P\,,\,-P\,,-p_2$	$-P$	p_1
G	$P\,,\,P\,,0$	P	$-P$
G	$p_1\,,\,p_1\,,\,-p_2$	p_1	$-P$
G	$p_2\,,\,p_2\,,\,-p_1$	p_2	$-P$
G	$-p_2\,,\,-p_2\,,\,p_1$	$-p_2$	P

Table 2: The basis $p_{\rm first}$, $p_{\rm second}$ for expanding form factors occurring in the reduction of tensor integrals corresponding to diagrams with a larger number of propagators. First entry is always the defyining representation. An example is given in Eq.(405).

$$D = p^2_{\rm first}\,p^2_{\rm second} - (p_{\rm first}\cdot p_{\rm second})^2, \qquad \text{etc}, \tag{406}$$

(see Eq.(9)) where, at the first level of reduction, we always have $p_{\rm first} = p_1$ and $p_{\rm second} = p_2$.

B.8 Reduction for rank three tensors

For rank three tensors the number of form factors and of contractions (tensors with saturated indices) increases considerably and it is not convenient to write down all cases explicitly; we prefer to adopt a different way of collecting the results. The reduction technique is based on two algorithms which we illustrate in the case of the V^M family.

A1. Contraction of tensor integrals with $\delta^{\mu\nu}$ and decomposition of tensors of lower rank

$$\begin{aligned}
&\delta^{\mu\nu}\int d^nq_1\int d^nq_2\,\frac{q_{2\mu}\,q_{2\nu}\,q_{2\alpha}}{[1][2]_M[3]_M[4]_M[5]_M[6]_M}\\
&=\int d^nq_1\int d^nq_2\,\frac{q_{2\alpha}}{[1][2]_M[4]_M[5]_M[6]_M} - m_3^2\int d^nq_1\int d^nq_2\,\frac{q_{2\alpha}}{[1][2]_M[3]_M[4]_M[5]_M[6]_M}\\
&=\frac{\pi^4}{\mu^{2\epsilon}}\sum_{i=1,2}\Big[V^{I}_{2i} - m_3^2\,V^{M}_{2i}\Big]\,p_{i\alpha} = \sum_{i=1,2} v^{M}_{222i}\,p_{i\alpha}.
\end{aligned} \tag{407}$$

A2. Contraction of tensor integrals with with p_1^μ or p_2^μ (or P^μ) and decomposition of tensors of lower rank: for instance we obtain

$$V^M(0\,|\,p_1,\nu,\alpha) = v^M_{2223}\,p_{1\nu}\,p_{1\alpha} + v^M_{2224}\,p_{2\nu}\,p_{2\alpha} + v^M_{2225}\,\{p_1p_2\}_{\nu\alpha} + v^M_{2226}\,\delta_{\nu\alpha}. \tag{408}$$

The v_i^M originate from the decomposition of a rank two tensor after using Eq.(173) and writing

$$2\,q_2\cdot p_1 = [4]_M - [3]_M - p_1^2 + m_4^2 - m_3^2, \tag{409}$$

and after shifting the loop momenta in order to recover the standard form of diagrams with fewer propagators. Therefore we have

$$\begin{aligned} V^M_{2224}\,p_1^2 + V^M_{2223}\,p_{12} + 2\,V^M_{2221} &= v^M_{2223}, \qquad & V^M_{2224}\,p_1^2 + V^M_{2226}\,p_{12} &= v^M_{2224}, \\ V^M_{2223}\,p_1^2 + V^M_{2224}\,p_{12} + V^M_{2222} &= v^M_{2225}, \qquad & V^M_{2221}\,p_1^2 + V^M_{2222}\,p_{12} &= v^M_{2226}. \end{aligned} \tag{410}$$

The choice of contractions is limited by the request that the resulting scalar products be reducible. In each case one obtains a system of equations for the rank three form factors to be solved in terms of lower rank form factors and of generalized scalars. Decompositions of vector integrals are defined in Eq.(118), Eq.(153), Eq.(174), Eq.(195), Eq.(223) and Eq.(252). Decompositions for rank two tensor integrals are defined in Eq.(138), for V^E, in Eq.(161) for V^I, in Eq.(180) for V^M, in Eq.(205), Eq.(210), Eq.(216) for V^G, in Eq.(232), Eq.(238), Eq.(243) for V^K and in Eq.(256), Eq.(260), Eq.(265) for V^H.

B.8.1 Contractions of rank three tensor integrals

In this Section we collect the results for all contractions of rank three tensors (with a Kronecker delta functions or with an external momentum) that give rise to reducible scalar products. These definitions will be used in Sects. B.8.3 – B.8.5 to construct tensors with three saturated indices and to build systems of equations that can be solved for the corresponding form factors. First we define the relevant contractions, once again those that are leading to reducible scalar products in the numerators:

- ***M* family**

$$\begin{aligned}
V^M(0\,|\,\mu,\mu,\nu) &= v^M_{2221}\,p_{1\nu} + v^M_{2222}\,p_{2\nu}, \\
V^M(0\,|\,p_1,\mu,\nu) &= v^M_{2223}\,p_{1\mu}\,p_{1\nu} + v^M_{2224}\,p_{2\mu}\,p_{2\nu} + v^M_{2225}\,\{p_1p_2\}_{\mu\nu} + v^M_{2226}\,\delta_{\mu\nu}, \\
V^M(0\,|\,p_2,\mu,\nu) &= v^M_{2227}\,p_{1\mu}\,p_{1\nu} + v^M_{2228}\,p_{2\mu}\,p_{2\nu} + v^M_{2229}\,\{p_1p_2\}_{\mu\nu} + v^M_{22210}\,\delta_{\mu\nu}, \\[1ex]
V^M(\nu\,|\,\mu,\mu) &= v^M_{1221}\,p_{1\nu} + v^M_{1222}\,p_{2\nu}, \\
V^M(\mu\,|\,p_1,\nu) &= v^M_{1223}\,p_{1\mu}\,p_{1\nu} + v^M_{1224}\,p_{2\mu}\,p_{2\nu} + v^M_{1225}\,\{p_1p_2\}_{\mu\nu} + v^M_{1226}\,\delta_{\mu\nu}, \\
V^M(\mu\,|\,p_2,\nu) &= v^M_{1227}\,p_{1\mu}\,p_{1\nu} + v^M_{1228}\,p_{2\mu}\,p_{2\nu} + v^M_{1229}\,\{p_1p_2\}_{\mu\nu} + v^M_{12210}\,\delta_{\mu\nu}, \\
V^M(\mu\,|\,\mu,\nu) &= v^M_{12211}\,p_{1\nu} + v^M_{12212}\,p_{2\nu}, \\[1ex]
V^M(\mu,\mu\,|\,\nu) &= v^M_{1121}\,p_{1\nu} + v^M_{1122}\,p_{2\nu}, \\
V^M(\mu,\nu\,|\,p_1) &= v^M_{1123}\,p_{1\mu}\,p_{1\nu} + v^M_{1124}\,p_{2\mu}\,p_{2\nu} + v^M_{1125}\,\{p_1p_2\}_{\mu\nu} + v^M_{1126}\,\delta_{\mu\nu}, \\
V^M(\mu,\nu\,|\,p_2) &= v^M_{1127}\,p_{1\mu}\,p_{1\nu} + v^M_{1128}\,p_{2\mu}\,p_{2\nu} + v^M_{1129}\,\{p_1p_2\}_{\mu\nu} + v^M_{11210}\,\delta_{\mu\nu}, \\
V^M(\mu,\nu\,|\,\mu) &= v^M_{11211}\,p_{1\nu} + v^M_{11212}\,p_{2\nu}.
\end{aligned}$$

$$V^M(\mu,\mu,\nu\,|\,0) = v^M_{1111}\,p_{1\nu} + v^M_{1112}\,p_{2\nu}, \tag{411}$$

- ***K* family**

$$
\begin{aligned}
V^K(0\,|\,\nu,\nu,\mu) &= v^K_{2221}\,p_{1\mu} + v^K_{2222}\,p_{2\mu},\\
V^K(0\,|\,p_1,\mu,\nu) &= v^K_{2223}\,p_{1\mu}\,p_{1\nu} + v^K_{2224}\,p_{2\mu}\,p_{2\nu} + v^K_{2225}\,\{p_1p_2\}_{\mu\nu} + v^K_{2226}\,\delta_{\mu\nu},\\
V^K(0\,|\,p_2,\mu,\nu) &= v^K_{2227}\,p_{1\mu}\,p_{1\nu} + v^K_{2228}\,p_{2\mu}\,p_{2\nu} + v^K_{2229}\,\{p_1p_2\}_{\mu\nu} + v^K_{22210}\,\delta_{\mu\nu},\\[1em]
V^K(\nu,\nu,\mu\,|\,0) &= v^K_{1111}\,p_{1\mu} + v^K_{1112}\,p_{2\mu},\\
V^K(P,\mu,\nu\,|\,0) &= v^K_{1113}\,p_{1\mu}\,p_{1\nu} + v^K_{1114}\,p_{2\mu}\,p_{2\nu} + v^K_{1115}\,\{p_1p_2\}_{\mu\nu} + v^K_{1116}\,\delta_{\mu\nu},\\[1em]
V^K(\nu,\nu\,|\,\mu) &= v^K_{1121}\,p_{1\mu} + v^K_{1122}\,p_{2\mu},\\
V^K(P,\mu\,|\,\nu) &= v^K_{1123}\,p_{1\mu}\,p_{1\nu} + v^K_{1124}\,p_{2\mu}\,p_{2\nu} + v^K_{1125}\,p_{1\mu}\,p_{2\nu} + v^K_{1126}\,p_{1\nu}\,p_{2\mu} + v^K_{1127}\,\delta_{\mu\nu},\\
V^K(\mu,\nu\,|\,p_1) &= v^K_{1128}\,p_{1\mu}\,p_{1\nu} + v^K_{1129}\,p_{2\mu}\,p_{2\nu} + v^K_{11210}\,\{p_1p_2\}_{\mu\nu} + v^K_{11211}\,\delta_{\mu\nu},\\
V^K(\mu,\nu\,|\,p_2) &= v^K_{11212}\,p_{1\mu}\,p_{1\nu} + v^K_{11213}\,p_{2\mu}\,p_{2\nu} + v^K_{11214}\,\{p_1p_2\}_{\mu\nu} + v^K_{11215}\,\delta_{\mu\nu},\\
V^K(\nu,\mu\,|\,\nu) &= v^K_{11216}\,p_{1\mu} + v^K_{11217}\,p_{2\mu},\\[1em]
V^K(\mu\,|\,\nu,\nu) &= v^K_{1221}\,p_{1\mu} + v^K_{1222}\,p_{2\mu},\\
V^K(P\,|\,\mu,\nu) &= v^K_{1223}\,p_{1\mu}\,p_{1\nu} + v^K_{1224}\,p_{2\mu}\,p_{2\nu} + v^K_{1225}\,\{p_1p_2\}_{\mu\nu} + v^K_{1226}\,\delta_{\mu\nu},\\
V^K(\mu\,|\,\nu,p_1) &= v^K_{1227}\,p_{1\mu}\,p_{1\nu} + v^K_{1228}\,p_{2\mu}\,p_{2\nu} + v^K_{1229}\,p_{1\mu}\,p_{2\nu} + v^K_{12210}\,p_{1\nu}\,p_{2\mu} + v^K_{12211}\,\delta_{\mu\nu},\\
V^K(\mu\,|\,\nu,p_2) &= v^K_{12212}\,p_{1\mu}\,p_{1\nu} + v^K_{12213}\,p_{2\mu}\,p_{2\nu} + v^K_{12214}\,p_{1\mu}\,p_{2\nu} + v^K_{12215}\,p_{1\nu}\,p_{2\mu} + v^K_{12216}\,\delta_{\mu\nu},\\
V^K(\nu\,|\,\nu,\mu) &= v^K_{12217}\,p_{1\mu} + v^K_{12218}\,p_{2\mu}.
\end{aligned}
\tag{412}
$$

- ***H*** **family**

$$
\begin{aligned}
V^H(0\,|\,\nu,\nu,\mu) &= v^H_{2221}\,p_{1\mu} + v^H_{2222}\,p_{2\mu},\\
V^H(0\,|\,p_1,\mu,\nu) &= v^H_{2223}\,p_{1\mu}\,p_{1\nu} + v^H_{2224}\,p_{2\mu}\,p_{2\nu} + v^H_{2225}\,\{p_1p_2\}_{\mu\nu} + v^H_{2226}\,\delta_{\mu\nu},\\[1em]
V^H(\nu,\nu,\mu\,|\,0) &= v^H_{1111}\,p_{1\mu} + v^H_{1112}\,p_{2\mu},\\
V^H(p_2,\mu,\nu\,|\,0) &= v^H_{1113}\,p_{1\mu}\,p_{1\nu} + v^H_{1114}\,p_{2\mu}\,p_{2\nu} + v^H_{1115}\,\{p_1p_2\}_{\mu\nu} + v^H_{1116}\,\delta_{\mu\nu},\\[1em]
V^H(\nu,\nu\,|\,\mu) &= v^H_{1121}\,p_{1\mu} + v^H_{1122}\,p_{2\mu},\\
V^H(p_2,\mu\,|\,\nu) &= v^H_{1123}\,p_{1\mu}\,p_{1\nu} + v^H_{1124}\,p_{2\mu}\,p_{2\nu} + v^H_{1125}\,p_{1\mu}\,p_{2\nu} + v^H_{1126}\,p_{1\nu}\,p_{2\mu} + v^H_{1127}\,\delta_{\mu\nu},\\
V^H(\mu,\nu\,|\,p_1) &= v^H_{1128}\,p_{1\mu}\,p_{1\nu} + v^H_{1129}\,p_{2\mu}\,p_{2\nu} + v^H_{11210}\,\{p_1p_2\}_{\mu\nu} + v^H_{11211}\,\delta_{\mu\nu},\\[1em]
V^H(\mu\,|\,\nu,\nu) &= v^H_{1221}\,p_{1\mu} + v^H_{1222}\,p_{2\mu},\\
V^H(p_2\,|\,\mu,\nu) &= v^H_{1223}\,p_{1\mu}\,p_{1\nu} + v^H_{1224}\,p_{2\mu}\,p_{2\nu} + v^H_{1225}\,\{p_1p_2\}_{\mu\nu} + v^H_{1226}\,\delta_{\mu\nu},\\
V^H(\mu\,|\,\nu,p_1) &= v^H_{1227}\,p_{1\mu}\,p_{1\nu} + v^H_{1228}\,p_{2\mu}\,p_{2\nu} + v^H_{1229}\,p_{1\mu}\,p_{2\nu} + v^H_{12210}\,p_{1\nu}\,p_{2\mu} + v^H_{12211}\,\delta_{\mu\nu}.
\end{aligned}
\tag{413}
$$

B.8.2 Evaluation of contracted rank three tensor integrals

Successively all contractions of Eqs.(411)–(413) are expressed as linear combinations of form factors of lower rank. These relations can be used as they stand or we can insert, recursively, results for rank two and rank one form factors (listed in Sects. B.1 – B.6) until one reaches a result which is written in terms of scalar integrals only.

- ***M*** **family** Eq.(411)

$$
v^M_{222i} = -m_3^2\,V^M_{2i} + V^I_{2i}(p_1,P,\{m\}_{12345}), \qquad i = 1,2
$$

$$
\begin{aligned}
v^{M}_{222\,i+2} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{22i} - V^{I}_{22i}(p_1,P,\{m\}_{12345}) + V^{I}_{222}(0,P,\{m\}_{12335})\Big], \qquad i = 1\cdots 3,\\
v^{M}_{2226} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{224} - V^{I}_{224}(p_1,P,\{m\}_{12345}) + V^{I}_{224}(0,P,\{m\}_{12335})\Big],\\
v^{M}_{2227} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{221} - V^{I}_{222}(0,P,\{m\}_{12335}) + V^{I}_{221}(p_1,0,\{m\}_{12343}) + V^{I}_{222}(p_1,0,\{m\}_{12343})\\
&\quad - 2\,V^{I}_{223}(p_1,P,\{m\}_{12343})\Big],\\
v^{M}_{222\,i+6} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{22i} - V^{I}_{222}(0,P,\{m\}_{12335})\Big], \quad i = 2,3,\\
v^{M}_{22210} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{224} + V^{I}_{224}(p_1,0,\{m\}_{12343}) - V^{I}_{224}(0,P,\{m\}_{12335})\Big],
\end{aligned} \tag{414}
$$

$$
v^{M}_{111i} = -m_1^2\,V^{M}_{1i} - A_0(m_2)\,C_{1i}(2,1,1\,;\,p_1,p_2,\{m\}_{345}), \qquad i = 1,2, \tag{415}
$$

$$
\begin{aligned}
v^{M}_{112i} &= -m_1^2\,V^{M}_{2i} - A_0(m_2)\,C_{1i}(2,1,1\,;\,p_1,p_2,\{m\}_{345}), \qquad i = 1,2,\\
v^{M}_{112\,i+2} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{11i} - V^{I}_{11i}(p_1,P,\{m\}_{12345}) + V^{I}_{112}(0,P,\{m\}_{12335})\Big], \quad i = 1\cdots 3,\\
v^{M}_{1126} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{114} - V^{I}_{114}(p_1,P,\{m\}_{12345}) + V^{I}_{114}(0,P,\{m\}_{12335})\Big],\\
v^{M}_{1127} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{111} - V^{I}_{112}(0,P,\{m\}_{12335}) + V^{I}_{111}(p_1,0,\{m\}_{12343}) + V^{I}_{112}(p_1,0,\{m\}_{12343})\\
&\quad - 2\,V^{I}_{111}(p_1,0,\{m\}_{12343})\Big],\\
v^{M}_{112\,i+6} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{11i} - V^{I}_{112}(0,P,\{m\}_{12335})\Big], \quad i = 2,3,\\
v^{M}_{11210} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{114} - V^{I}_{114}(0,P,\{m\}_{12335}) + V^{I}_{114}(p_1,0,\{m\}_{12343})\Big],\\
v^{M}_{112\,i+10} &= \frac{1}{2}\Big[-m_{123}^2\,V^{M}_{1i} + V^{I}_{1i}(p_1,P,\{m\}_{12345}) - A_0(m_2)\,C_{1i}(2,1,1\,;\,p_1,p_2,\{m\}_{345})\Big], \quad i = 1,2,
\end{aligned} \tag{416}
$$

$$
\begin{aligned}
v^{M}_{122i} &= -m_3^2\,V^{M}_{1i} + V^{I}_{1i}(p_1,P,\{m\}_{12345}), \qquad i = 1,2,\\
v^{M}_{122\,i+2} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{12i} - V^{I}_{12i}(p_1,P,\{m\}_{12345}) + V^{I}_{122}(0,P,\{m\}_{12335})\Big], \quad i = 1\cdots 3,\\
v^{M}_{1226} &= \frac{1}{2}\Big[-l_{134}\,V^{M}_{124} - V^{I}_{124}(p_1,P,\{m\}_{12345}) + V^{I}_{124}(0,P,\{m\}_{12335})\Big],\\
v^{M}_{1227} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{121} - V^{I}_{122}(0,P,\{m\}_{12335}) + V^{I}_{121}(p_1,0,\{m\}_{12343}) + V^{I}_{122}(p_1,0,\{m\}_{12343})\\
&\quad - 2\,V^{I}_{123}(p_1,0,\{m\}_{12343})\Big],\\
v^{M}_{122\,i+6} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{12i} - V^{I}_{122}(0,P,\{m\}_{12335})\Big], \quad i = 2,3,\\
v^{M}_{12210} &= \frac{1}{2}\Big[(l_{134}-l_{P}35)\,V^{M}_{124} + V^{I}_{124}(p_1,0,\{m\}_{12343}) - V^{I}_{124}(0,P,\{m\}_{12335})\Big],\\
v^{M}_{122\,i+10} &= \frac{1}{2}\Big[m_{123}^2\,V^{M}_{2i} + V^{I}_{2i}(p_1,P,\{m\}_{12345}) + A_0([m_1,m_2])\,C_{1i}(2,1,1\,;\,p_1,p_2,\{m\}_{345})\Big], \quad i = 1,2
\end{aligned} \tag{417}
$$

Furthermore we define

$$
\begin{aligned}
v^{M}_{1113} &= 36\,\omega^6\,V^{1|2,4,1|4}_{M}\Big|_{n=10-\epsilon}, \qquad v^{M}_{1114} = 12\,\omega^6\,V^{1|2,3,2|4}_{M}\Big|_{n=10-\epsilon},\\
v^{M}_{1115} &= 12\,\omega^6\,V^{1|2,2,3|4}_{M}\Big|_{n=10-\epsilon}, \qquad v^{M}_{1116} = 36\,\omega^6\,V^{1|2,1,4|4}_{M}\Big|_{n=10-\epsilon}.
\end{aligned} \tag{418}
$$

• ***K* family** Eq.(412)

$$
\begin{aligned}
v^{K}_{2221} &= -V^{K}_{21}\, m_4^2 + V^{G}_{21}(P,P,p_1,\{m\}_{12365}),\\
v^{K}_{2222} &= -V^{K}_{22}\, m_4^2 + V^{G}_{21}(P,P,p_1,\{m\}_{12365}) - V^{G}_{22}(P,P,p_1,\{m\}_{12365}),\\
v^{K}_{2223} &= \frac{1}{2}\Big[-V^{K}_{221}\, l_{145} - V^{G}_{221}(P,P,p_1,\{m\}_{12365}) + V^{G}_{221}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{222}(P,P,0,\{m\}_{12364}) - 2\,V^{G}_{223}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{2224} &= \frac{1}{2}\Big[-V^{K}_{222}\, l_{145} - V^{G}_{221}(P,P,p_1,\{m\}_{12365}) + V^{G}_{221}(P,P,0,\{m\}_{12364})\\
&\quad - V^{G}_{222}(P,P,p_1,\{m\}_{12365}) + V^{G}_{222}(P,P,0,\{m\}_{12364}) + 2\,V^{G}_{223}(P,P,p_1,\{m\}_{12365})\\
&\quad - 2\,V^{G}_{223}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{2225} &= \frac{1}{2}\Big[-V^{K}_{223}\, l_{145} - V^{G}_{221}(P,P,p_1,\{m\}_{12365}) + V^{G}_{221}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{222}(P,P,0,\{m\}_{12364}) + V^{G}_{223}(P,P,p_1,\{m\}_{12365}) - 2\,V^{G}_{223}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{2226} &= \frac{1}{2}\Big[-V^{K}_{22}4\, l_{145} - V^{G}_{224}(P,P,p_1,\{m\}_{12365}) + V^{G}_{224}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{2227} &= \frac{1}{2}\Big[V^{K}_{221}\,(l_{165} - P^2) - V^{G}_{221}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{221}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{222}(P,P,0,\{m\}_{12364}) + V^{G}_{222}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + 2\,V^{G}_{223}(P,P,0,\{m\}_{12364}) - 2\,V^{G}_{223}(-P,-P,-p_2,\{m\}_{21345}) + 2\,V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - 2\,V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{K}_{2228} &= \frac{1}{2}\Big[V^{K}_{222}\,(l_{165} - P^2) - V^{G}_{221}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{221}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{222}(P,P,0,\{m\}_{12364}) + 2\,V^{G}_{223}(P,P,0,\{m\}_{12364})\\
&\quad + 2\,V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{K}_{2229} &= \frac{1}{2}\Big[V^{K}_{223}\,(l_{165} - P^2) - V^{G}_{221}(P,P,0,\{m\}_{12364}) + V^{G}_{221}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - V^{G}_{222}(P,P,0,\{m\}_{12364}) + 2\,V^{G}_{223}(P,P,0,\{m\}_{12364}) - V^{G}_{223}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + 2\,V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{K}_{22210} &= \frac{1}{2}\Big[V^{K}_{224}\,(l_{165} - P^2) - V^{G}_{224}(P,P,0,\{m\}_{12364}) + V^{G}_{224}(-P,-P,-p_2,\{m\}_{21345})\Big],
\end{aligned}
\tag{419}
$$

$$
\begin{aligned}
v^{K}_{1111} &= -V^{K}_{11}\, m_1^2 - V^{I}_{12}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654}),\\
v^{K}_{1112} &= -V^{K}_{12}\, m_1^2 - V^{I}_{11}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654}),\\
v^{K}_{1113} &= \frac{1}{2}\Big[-V^{K}_{111}\, l_{P12} + V^{I}_{111}(p_1,P,\{m\}_{13456}) - V^{I}_{112}(-p_2,-P,\{m\}_{23654})\\
&\quad - 2\,V^{I}_{12}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1114} &= \frac{1}{2}\Big[-V^{K}_{112}\, l_{P12} - V^{I}_{111}(-p_2,-P,\{m\}_{23654}) + V^{I}_{112}(p_1,P,\{m\}_{13456})\\
&\quad - 2\,V^{I}_{11}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1115} &= \frac{1}{2}\Big[-V^{K}_{113}\, l_{P12} + V^{I}_{113}(p_1,P,\{m\}_{13456}) - V^{I}_{113}(-p_2,-P,\{m\}_{23654})\\
&\quad - V^{I}_{11}(-p_2,-P,\{m\}_{23654}) - V^{I}_{12}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],
\end{aligned}
$$

$$v^{K}_{1116} = \frac{1}{2}\Big[-V^{K}_{114}\, l_{P}12 + V^{I}_{114}(p_1, P, \{m\}_{13456}) - V^{I}_{114}(-p_2, -P, \{m\}_{23654})\Big], \tag{420}$$

$$\begin{aligned}
v^{K}_{1121} &= -V^{K}_{21}\, m_1^2 - V^{I}_{22}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654}),\\
v^{K}_{1122} &= -V^{K}_{22}\, m_1^2 - V^{I}_{21}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654}),\\
v^{K}_{1123} &= \frac{1}{2}\Big[-V^{K}_{121}\, l_{P}12 + V^{I}_{121}(p_1, P, \{m\}_{13456}) - V^{I}_{122}(-p_2, -P, \{m\}_{23654})\\
&\quad - V^{I}_{22}(-p_2, -P, \{m\}_{23654}) - V^{I}_{12}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654})\Big],\\
v^{K}_{1124} &= \frac{1}{2}\Big[-V^{K}_{122}\, l_{P}12 - V^{I}_{121}(-p_2, -P, \{m\}_{23654}) + V^{I}_{122}(p_1, P, \{m\}_{13456})\\
&\quad - V^{I}_{21}(-p_2, -P, \{m\}_{23654}) - V^{I}_{11}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654})\Big],\\
v^{K}_{1125} &= \frac{1}{2}\Big[-V^{K}_{123}\, l_{P}12 + V^{I}_{123}(p_1, P, \{m\}_{13456}) - V^{I}_{125}(-p_2, -P, \{m\}_{23654})\\
&\quad - V^{I}_{21}(-p_2, -P, \{m\}_{23654}) - V^{I}_{12}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654})\Big],\\
v^{K}_{1126} &= \frac{1}{2}\Big[-V^{K}_{125}\, l_{P}12 - V^{I}_{123}(-p_2, -P, \{m\}_{23654}) + V^{I}_{125}(p_1, P, \{m\}_{13456})\\
&\quad - V^{I}_{22}(-p_2, -P, \{m\}_{23654}) - V^{I}_{11}(-p_2, -P, \{m\}_{23654}) - V^{I}_{0}(-p_2, -P, \{m\}_{23654})\Big],\\
v^{K}_{1127} &= \frac{1}{2}\Big[-V^{K}_{124}\, l_{P}12 + V^{I}_{124}(p_1, P, \{m\}_{13456}) - V^{I}_{124}(-p_2, -P, \{m\}_{23654})\Big],\\
v^{K}_{1128} &= \frac{1}{2}\Big[-V^{K}_{111}\, l_{145} - V^{G}_{111}(P, P, p_1, \{m\}_{12365}) + V^{G}_{111}(P, P, 0, \{m\}_{12364})\\
&\quad + V^{G}_{112}(P, P, 0, \{m\}_{12364}) - 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364})\Big],\\
v^{K}_{1129} &= \frac{1}{2}\Big[-V^{K}_{112}\, l_{145} - V^{G}_{111}(P, P, p_1, \{m\}_{12365}) + V^{G}_{111}(P, P, 0, \{m\}_{12364})\\
&\quad - V^{G}_{112}(P, P, p_1, \{m\}_{12365}) + V^{G}_{112}(P, P, 0, \{m\}_{12364}) + 2\, V^{G}_{113}(P, P, p_1, \{m\}_{12365})\\
&\quad - 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364})\Big],\\
v^{K}_{11210} &= \frac{1}{2}\Big[-V^{K}_{113}\, l_{145} - V^{G}_{111}(P, P, p_1, \{m\}_{12365}) + V^{G}_{111}(P, P, 0, \{m\}_{12364})\\
&\quad + V^{G}_{112}(P, P, 0, \{m\}_{12364}) + V^{G}_{113}(P, P, p_1, \{m\}_{12365}) - 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364})\Big],\\
v^{K}_{11211} &= \frac{1}{2}\Big[-V^{K}_{114}\, l_{145} - V^{G}_{114}(P, P, p_1, \{m\}_{12365}) + V^{G}_{114}(P, P, 0, \{m\}_{12364})\Big],\\
v^{K}_{11212} &= \frac{1}{2}\Big[V^{K}_{111}\,(l_{165} - P^2) - V^{G}_{111}(P, P, 0, \{m\}_{12364}) + V^{G}_{111}(-P, -P, -p_2, \{m\}_{21345})\\
&\quad - V^{G}_{112}(P, P, 0, \{m\}_{12364}) + V^{G}_{112}(-P, -P, -p_2, \{m\}_{21345}) + 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364})\\
&\quad - 2\, V^{G}_{113}(-P, -P, -p_2, \{m\}_{21345}) + 2\, V^{G}_{11}(-P, -P, -p_2, \{m\}_{21345}) - 2\, V^{G}_{12}(-P, -P, -p_2, \{m\}_{21345})\\
&\quad + V^{G}_{0}(-P, -P, -p_2, \{m\}_{21345})\Big],\\
v^{K}_{11213} &= \frac{1}{2}\Big[V^{K}_{112}\,(l_{165} - P^2) - V^{G}_{111}(P, P, 0, \{m\}_{12364}) + V^{G}_{111}(-P, -P, -p_2, \{m\}_{21345})\\
&\quad - V^{G}_{112}(P, P, 0, \{m\}_{12364}) + 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364}) + 2\, V^{G}_{11}(-P, -P, -p_2, \{m\}_{21345})\\
&\quad + V^{G}_{0}(-P, -P, -p_2, \{m\}_{21345})\Big],\\
v^{K}_{11214} &= \frac{1}{2}\Big[V^{K}_{113}\,(l_{165} - P^2) - V^{G}_{111}(P, P, 0, \{m\}_{12364}) + V^{G}_{111}(-P, -P, -p_2, \{m\}_{21345})\\
&\quad - V^{G}_{112}(P, P, 0, \{m\}_{12364}) + 2\, V^{G}_{113}(P, P, 0, \{m\}_{12364}) - V^{G}_{113}(-P, -P, -p_2, \{m\}_{21345})
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\, V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{K}_{11215} &= \frac{1}{2}\Big[V^{K}_{114}\,(l_{165}-P^2) - V^{G}_{114}(P,P,0,\{m\}_{12364}) + V^{G}_{114}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{K}_{11216} &= \frac{1}{2}\Big[-V^{K}_{11}\,m^2_{134} + V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{I}_{12}(-p_2,-P,\{m\}_{23654})\\
&\quad + B_1(P,\{m\}_{12})\,C_0(p_1,p_2,\{m\}_{456}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{11217} &= \frac{1}{2}\Big[-V^{K}_{12}\,m^2_{134} + V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{G}_{12}(P,P,p_1,\{m\}_{12365})\\
&\quad - V^{I}_{11}(-p_2,-P,\{m\}_{23654}) + B_1(P,\{m\}_{12})\,C_0(p_1,p_2,\{m\}_{456}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],
\end{aligned}
\tag{421}
$$

$$
\begin{aligned}
v^{K}_{1221} &= -V^{K}_{11}\,m^2_{4} + V^{G}_{11}(P,P,p_1,\{m\}_{12365}),\\
v^{K}_{1222} &= -V^{K}_{12}\,m^2_{4} + V^{G}_{11}(P,P,p_1,\{m\}_{12365}) - V^{G}_{12}(P,P,p_1,\{m\}_{12365}),\\
v^{K}_{1223} &= \frac{1}{2}\Big[-V^{K}_{221}\,l_{P12} + V^{I}_{221}(p_1,P,\{m\}_{13456}) - V^{I}_{222}(-p_2,-P,\{m\}_{23654})\\
&\quad - 2\,V^{I}_{22}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1224} &= \frac{1}{2}\Big[-V^{K}_{222}\,l_{P12} - V^{I}_{221}(-p_2,-P,\{m\}_{23654}) + V^{I}_{222}(p_1,P,\{m\}_{13456})\\
&\quad - 2\,V^{I}_{21}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1225} &= \frac{1}{2}\Big[-V^{K}_{223}\,l_{P12} + V^{I}_{223}(p_1,P,\{m\}_{13456}) - V^{I}_{223}(-p_2,-P,\{m\}_{23654})\\
&\quad - V^{I}_{21}(-p_2,-P,\{m\}_{23654}) - V^{I}_{22}(-p_2,-P,\{m\}_{23654}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1226} &= \frac{1}{2}\Big[-V^{K}_{224}\,l_{P12} + V^{I}_{224}(p_1,P,\{m\}_{13456}) - V^{I}_{224}(-p_2,-P,\{m\}_{23654})\Big],\\
v^{K}_{1227} &= \frac{1}{2}\Big[-V^{K}_{121}\,l_{145} - V^{G}_{121}(P,P,p_1,\{m\}_{12365}) + V^{G}_{121}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{122}(P,P,0,\{m\}_{12364}) - V^{G}_{123}(P,P,0,\{m\}_{12364}) - V^{G}_{125}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{1228} &= \frac{1}{2}\Big[-V^{K}_{122}\,l_{145} - V^{G}_{121}(P,P,p_1,\{m\}_{12365}) + V^{G}_{121}(P,P,0,\{m\}_{12364})\\
&\quad - V^{G}_{122}(P,P,p_1,\{m\}_{12365}) + V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{123}(P,P,p_1,\{m\}_{12365})\\
&\quad - V^{G}_{123}(P,P,0,\{m\}_{12364}) + V^{G}_{125}(P,P,p_1,\{m\}_{12365}) - V^{G}_{125}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{1229} &= \frac{1}{2}\Big[-V^{K}_{123}\,l_{145} - V^{G}_{121}(P,P,p_1,\{m\}_{12365}) + V^{G}_{121}(P,P,0,\{m\}_{12364})\\
&\quad + V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{123}(P,P,p_1,\{m\}_{12365}) - V^{G}_{123}(P,P,0,\{m\}_{12364})\\
&\quad - V^{G}_{125}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{12210} &= \frac{1}{2}\Big[-V^{K}_{125}\,l_{145} - V^{G}_{121}(P,P,p_1,\{m\}_{12365}) + V^{G}_{121}(P,P,0,\{m\}_{12364}) + V^{G}_{122}(P,P,0,\{m\}_{12364})\\
&\quad - V^{G}_{123}(P,P,0,\{m\}_{12364}) + V^{G}_{125}(P,P,p_1,\{m\}_{12365}) - V^{G}_{125}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{12211} &= \frac{1}{2}\Big[-V^{K}_{124}\,l_{145} - V^{G}_{124}(P,P,p_1,\{m\}_{12365}) + V^{G}_{124}(P,P,0,\{m\}_{12364})\Big],\\
v^{K}_{12212} &= \frac{1}{2}\Big[V^{K}_{121}\,(l_{165}-P^2) - V^{G}_{121}(P,P,0,\{m\}_{12364}) + V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{122}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{123}(P,P,0,\{m\}_{12364})\\
&\quad - V^{G}_{123}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{125}(P,P,0,\{m\}_{12364}) - V^{G}_{125}(-P,-P,-p_2,\{m\}_{21345})
\end{aligned}
$$

$$
\begin{aligned}
&+ V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{K}_{12213} &= \frac{1}{2}\Big[V^{K}_{122}\,(l_{165}-P^2) - V^{G}_{121}(P,P,0,\{m\}_{12364}) + V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{123}(P,P,0,\{m\}_{12364}) + V^{G}_{125}(P,P,0,\{m\}_{12364}) \\
&+ V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{K}_{12214} &= \frac{1}{2}\Big[V^{K}_{123}\,(l_{165}-P^2) - V^{G}_{121}(P,P,0,\{m\}_{12364}) + V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{123}(P,P,0,\{m\}_{12364}) + V^{G}_{125}(P,P,0,\{m\}_{12364}) \\
&- V^{G}_{125}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{12}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{K}_{12215} &= \frac{1}{2}\Big[V^{K}_{125}\,(l_{165}-P^2) - V^{G}_{121}(P,P,0,\{m\}_{12364}) + V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&- V^{G}_{122}(P,P,0,\{m\}_{12364}) + V^{G}_{123}(P,P,0,\{m\}_{12364}) - V^{G}_{123}(-P,-P,-p_2,\{m\}_{21345}) \\
&+ V^{G}_{125}(P,P,0,\{m\}_{12364}) + V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) \\
&+ V^{G}_{11}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{K}_{12216} &= \frac{1}{2}\Big[V^{K}_{124}\,(l_{165}-P^2) - V^{G}_{124}(P,P,0,\{m\}_{12364}) + V^{G}_{124}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{K}_{12217} &= \frac{1}{2}\Big[-V^{K}_{21}\,m^2_{134} + V^{G}_{21}(P,P,p_1,\{m\}_{12365}) - V^{I}_{22}(-p_2,-P,\{m\}_{23654}) \\
&+ B_0(P,\{m\}_{12})\,C_{11}(p_1,p_2,\{m\}_{456}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big], \\
v^{K}_{12218} &= \frac{1}{2}\Big[-V^{K}_{22}\,m^2_{134} + V^{G}_{21}(P,P,p_1,\{m\}_{12365}) - V^{G}_{22}(P,P,p_1,\{m\}_{12365}) \\
&- V^{I}_{21}(-p_2,-P,\{m\}_{23654}) + B_0(P,\{m\}_{12})\,C_{12}(p_1,p_2,\{m\}_{456}) - V^{I}_{0}(-p_2,-P,\{m\}_{23654})\Big].
\end{aligned}
\tag{422}
$$

- ***H* family** Eq.(412)

It is convenient to define certain combinations of form factors to be used in this family (they only appear in the present subsection):

$$
\begin{aligned}
&V^{I}_{Ai} = V^{I}_{11i} + V^{I}_{22i} - 2\,V^{I}_{12i}, \quad V^{I}_{Bi} = V^{I}_{1i} + V^{I}_{2i}, \quad V^{G}_{Ai} = V^{G}_{11i} - V^{G}_{12i}, \quad V^{G}_{Bi} = V^{G}_{22i} - V^{G}_{12i}, \\
&V^{G}_{Ci} = V^{G}_{1i} - V^{G}_{2i}, \quad V^{G}_{12A} = V^{G}_{125} - V^{G}_{123}.
\end{aligned}
\tag{423}
$$

We obtain

$$
\begin{aligned}
v^{H}_{2221} &= -V^{H}_{21}\,m^2_5 + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}), \\
v^{H}_{2222} &= -V^{H}_{22}\,m^2_5 + V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}), \\
v^{H}_{2223} &= \frac{1}{2}\Big[V^{H}_{221}\,(2\,p^2_1 - l_{165}) + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&- V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{B2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{B2}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&- 2\,V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634})\Big], \\
v^{H}_{2224} &= \frac{1}{2}\Big[V^{H}_{222}\,(2\,p^2_1 - l_{165}) + V^{G}_{A1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A1}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&+ V^{G}_{B1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{B1}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&- V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{B2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{B2}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&- 2\,V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + 2\,V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543}) - 2\,V^{G}_{B3}(p_2,p_2,-p_1,\{m\}_{21634})
\end{aligned}
$$

$$+ 2\,V^{G}_{B3}(-p_2,-p_2,p_1,\{m\}_{12543}) + 2\,V^{G}_{12A}(p_2,p_2,-p_1,\{m\}_{21634}) - 2\,V^{G}_{12A}(-p_2,-p_2,p_1,\{m\}_{12543})\Big],$$

$$\begin{aligned}
v^{H}_{2225} = \frac{1}{2}\Big[&V^{H}_{223}\,(2\,p_1^2 - l_{165}) + V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634})\\
&- V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{B2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{B2}(-p_2,-p_2,p_1,\{m\}_{12543})\\
&- V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543})\\
&- V^{G}_{B3}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{B3}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{12A}(p_2,p_2,-p_1,\{m\}_{21634})\\
&- V^{G}_{12A}(-p_2,-p_2,p_1,\{m\}_{12543})\Big],\\
v^{H}_{2226} = \frac{1}{2}\Big[&V^{H}_{224}\,(2\,p_1^2 - l_{165}) + V^{G}_{A4}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A4}(-p_2,-p_2,p_1,\{m\}_{12543})\\
&+ V^{G}_{B4}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{B4}(-p_2,-p_2,p_1,\{m\}_{12543})\Big],
\end{aligned} \tag{424}$$

$$\begin{aligned}
v^{H}_{1111} &= -V^{H}_{11}\,m_1^2 - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}) + V^{G}_{C2}(P,P,p_2,\{m\}_{34256}),\\
v^{H}_{1112} &= -V^{H}_{12}\,m_1^2 + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}),\\
v^{H}_{1113} &= \frac{1}{2}\Big[V^{H}_{111}\,l_{212} + V^{G}_{A1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{A1}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{A2}(P,P,p_2,\{m\}_{34256})\\
&\quad - V^{G}_{A2}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{B2}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B2}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - 2\,V^{G}_{A3}(P,P,p_2,\{m\}_{34256}) + 2\,V^{G}_{A3}(-P,-P,-p_2,\{m\}_{21345}) - 2\,V^{G}_{B3}(P,P,p_2,\{m\}_{34256})\\
&\quad + 2\,V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345}) + 2\,V^{G}_{12A}(P,P,p_2,\{m\}_{34256}) - 2\,V^{G}_{12A}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{H}_{1114} &= \frac{1}{2}\Big[V^{H}_{112}\,l_{212} + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) + V^{G}_{A1}(P,P,p_2,\{m\}_{34256})\\
&\quad - V^{G}_{A1}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - 2\,V^{G}_{C1}(P,P,p_2,\{m\}_{34256})\Big],\\
v^{H}_{1115} &= \frac{1}{2}\Big[V^{H}_{113}\,l_{212} + V^{G}_{A1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{A1}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{C1}(P,P,p_2,\{m\}_{34256})\\
&\quad + V^{G}_{C2}(P,P,p_2,\{m\}_{34256}) - V^{G}_{A3}(P,P,p_2,\{m\}_{34256}) + V^{G}_{A3}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{12A}(P,P,p_2,\{m\}_{34256})\\
&\quad - V^{G}_{12A}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{H}_{1116} &= \frac{1}{2}\Big[V^{H}_{114}\,l_{212} + V^{G}_{A4}(P,P,p_2,\{m\}_{34256}) - V^{G}_{A4}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{B4}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B4}(-P,-P,-p_2,\{m\}_{21345})\Big],
\end{aligned} \tag{425}$$

$$\begin{aligned}
v^{H}_{1121} &= -V^{H}_{21}\,m_1^2 + V^{G}_{21}(P,P,p_2,\{m\}_{34256}) - V^{G}_{22}(P,P,p_2,\{m\}_{34256}) + V^{G}_{0}(P,P,p_2,\{m\}_{34256}),\\
v^{H}_{1122} &= -V^{H}_{22}\,m_1^2 + V^{G}_{21}(P,P,p_2,\{m\}_{34256}) + V^{G}_{0}(P,P,p_2,\{m\}_{34256}),\\
v^{H}_{1123} &= \frac{1}{2}\Big[V^{H}_{121}\,l_{212} + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B2}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B2}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{C2}(P,P,p_2,\{m\}_{34256}) - 2\,V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) + 2\,V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345})\\
&\quad + V^{G}_{12A}(P,P,p_2,\{m\}_{34256}) - V^{G}_{12A}(-P,-P,-p_2,\{m\}_{21345})\Big],\\
v^{H}_{1124} &= \frac{1}{2}\Big[V^{H}_{122}\,l_{212} + V^{G}_{21}(P,P,p_2,\{m\}_{34256}) + V^{G}_{0}(P,P,p_2,\{m\}_{34256})
\end{aligned}$$

$$
\begin{aligned}
&\quad + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}) \\
&\quad + V^{G}_{C1}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1125} &= \frac{1}{2}\Big[V^{H}_{123}\, l_{212} + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}) + V^{G}_{C1}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{C2}(P,P,p_2,\{m\}_{34256}) \\
&\quad - V^{G}_{C2}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + V^{G}_{12A}(P,P,p_2,\{m\}_{34256}) - V^{G}_{12A}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1126} &= \frac{1}{2}\Big[V^{H}_{125}\, l_{212} + V^{G}_{21}(P,P,p_2,\{m\}_{34256}) - V^{G}_{22}(P,P,p_2,\{m\}_{34256}) \\
&\quad + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad - V^{G}_{C1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1127} &= \frac{1}{2}\Big[V^{H}_{124}\, l_{212} + V^{G}_{B4}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B4}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1128} &= \frac{1}{2}\Big[V^{H}_{111}\, (2\,p_1^2 - l_{165}) + V^{G}_{122}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{122}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{1129} &= \frac{1}{2}\Big[V^{H}_{112}\, (2\,p_1^2 - l_{165}) + V^{G}_{121}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{121}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + V^{G}_{122}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{122}(-p_2,-p_2,p_1,\{m\}_{12543}) - 2\,V^{G}_{123}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + 2\,V^{G}_{123}(-p_2,-p_2,p_1,\{m\}_{12543}) + 2\,V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) - 2\,V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A1}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + 2\,V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad - 2\,V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}) - 2\,V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + 2\,V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{11210} &= \frac{1}{2}\Big[V^{H}_{113}\, (2\,p_1^2 - l_{165}) + V^{G}_{122}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{122}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad - V^{G}_{123}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{123}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad - V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{11211} &= \frac{1}{2}\Big[V^{H}_{114}\, (2\,p_1^2 - l_{165}) + V^{G}_{124}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{124}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + V^{G}_{A4}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A4}(-p_2,-p_2,p_1,\{m\}_{12543})\Big],
\end{aligned} \tag{426}
$$

$$
\begin{aligned}
v^{H}_{1221} &= -V^{H}_{11}\, m_5^2 - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}), \\
v^{H}_{1222} &= -V^{H}_{12}\, m_5^2 + V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}), \\
v^{H}_{1223} &= \frac{1}{2}\Big[V^{H}_{221}\, l_{212} + V^{G}_{121}(P,P,p_2,\{m\}_{34256}) - V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + V^{G}_{122}(P,P,p_2,\{m\}_{34256}) - V^{G}_{122}(-P,-P,-p_2,\{m\}_{21345}) - 2\,V^{G}_{123}(P,P,p_2,\{m\}_{34256}) \\
&\quad + 2\,V^{G}_{123}(-P,-P,-p_2,\{m\}_{21345}) + 2\,V^{G}_{21}(P,P,p_2,\{m\}_{34256}) - 2\,V^{G}_{22}(P,P,p_2,\{m\}_{34256}) \\
&\quad + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + V^{G}_{B2}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B2}(-P,-P,-p_2,\{m\}_{21345}) - 2\,V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) \\
&\quad + 2\,V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345})\Big],
\end{aligned}
$$

$$
\begin{aligned}
v^{H}_{1224} &= \frac{1}{2}\Big[V^{H}_{222}\,l_{212} + V^{G}_{121}(P,P,p_2,\{m\}_{34256}) - V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + 2\,V^{G}_{21}(P,P,p_2,\{m\}_{34256}) - 2\,V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) \\
&\quad - V^{G}_{0}(-P,-P,-p_2,\{m\}_{21345}) + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1225} &= \frac{1}{2}\Big[V^{H}_{223}\,l_{212} + V^{G}_{121}(P,P,p_2,\{m\}_{34256}) - V^{G}_{121}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad - V^{G}_{123}(P,P,p_2,\{m\}_{34256}) + V^{G}_{123}(-P,-P,-p_2,\{m\}_{21345}) + 2\,V^{G}_{21}(P,P,p_2,\{m\}_{34256}) \\
&\quad - V^{G}_{21}(-P,-P,-p_2,\{m\}_{21345}) - V^{G}_{22}(P,P,p_2,\{m\}_{34256}) + V^{G}_{22}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + V^{G}_{0}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B1}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B1}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad - V^{G}_{B3}(P,P,p_2,\{m\}_{34256}) + V^{G}_{B3}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1226} &= \frac{1}{2}\Big[V^{H}_{224}\,l_{212} + V^{G}_{124}(P,P,p_2,\{m\}_{34256}) - V^{G}_{124}(-P,-P,-p_2,\{m\}_{21345}) \\
&\quad + V^{G}_{B4}(P,P,p_2,\{m\}_{34256}) - V^{G}_{B4}(-P,-P,-p_2,\{m\}_{21345})\Big], \\
v^{H}_{1227} &= \frac{1}{2}\Big[V^{H}_{121}\,(2\,p_1^2 - l_{165}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634})\Big], \\
v^{H}_{1228} &= \frac{1}{2}\Big[V^{H}_{122}\,(2\,p_1^2 - l_{165}) + V^{G}_{A1}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A1}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad - V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}) - 2\,V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + 2\,V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad + V^{G}_{12A}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{12A}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{1229} &= \frac{1}{2}\Big[V^{H}_{123}\,(2\,p_1^2 - l_{165}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) \\
&\quad - V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543}) + V^{G}_{12A}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad - V^{G}_{12A}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{12210} &= \frac{1}{2}\Big[V^{H}_{125}\,(2\,p_1^2 - l_{165}) + V^{G}_{21}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{22}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + V^{G}_{0}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{C1}(p_2,p_2,-p_1,\{m\}_{21634}) + V^{G}_{A2}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad - V^{G}_{A2}(-p_2,-p_2,p_1,\{m\}_{12543}) - 2\,V^{G}_{C2}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A3}(p_2,p_2,-p_1,\{m\}_{21634}) \\
&\quad + V^{G}_{A3}(-p_2,-p_2,p_1,\{m\}_{12543})\Big], \\
v^{H}_{12211} &= \frac{1}{2}\Big[V^{H}_{124}\,(2\,p_1^2 - l_{165}) + V^{G}_{A4}(p_2,p_2,-p_1,\{m\}_{21634}) - V^{G}_{A4}(-p_2,-p_2,p_1,\{m\}_{12543})\Big].
\end{aligned}
\tag{427}
$$

In the following Appendices results for rank three tensors with completely saturated indices are presented. At the same time we give an explicit solution for the form factors which are needed, implicitly, for some of the contracted expressions and, explicitly, for testing WST identities. We recall that tensors with saturated indices are of the upmost importance for applications related to projection techniques (see Section 4).

B.8.3 The V^M family

Almost all contracted tensors in this family can be trivially obtained using Eq.(411) and the definitions of Eqs.(414)–(417). Only for the 111 group we have to solve first for the form factors and then to replace the resulting expressions into the decomposition of the saturated tensors; in this way we are able to obtain

$$
V^{M}(\mu,\mu,p_1\,|\,0) = v^{M}_{1111}\,p_1^2 + v^{M}_{1112}\,p_{12}, \quad V^{M}(\mu,\mu,p_2\,|\,0) = v^{M}_{1111}\,p_{12} + v^{M}_{1112}\,p_2^2,
\tag{428}
$$

$$
\begin{aligned}
&(n+2)\,V^M(p_1,p_1,p_2\,|\,0) = 3\,v^M_{1111}\,D_3 - v^M_{1112}\,(2\,D - 3\,D_1) - v^M_{1113}\,p_1^2\,(n-1)\,D_3 \\
&+ v^M_{1116}\,p_2^2\,\Big[n\,D - (n-1)\,D_1\Big] - v^M_{1116}\,p_1^2\,\Big[6\,(n-2)\,D - (n-1)\,(9\,D_1 - 11\,D_3)\Big] \\
&+ \Big[v^M_{1116} + v^M_{1115}\Big]\,p_{12}\,\Big[(n-2)\,D - 3\,(n-1)\,D_1\Big] \\
&+ v^M_{1114}\,p_1^2\,\Big[2\,(n-2)\,D - 3\,(n-1)\,(D_1 - D_3)\Big], \\
&(n+2)\,V^M(p_2,p_2,p_1\,|\,0) = -v^M_{1111}\,(2\,D - 3\,D_1) + 3\,v^M_{1112}\,D_2 \\
&+ v^M_{1116}\,p_2^2\,\Big[2\,(n-2)\,D - (n-1)\,(3\,D_1 + D_2)\Big] \\
&+ 11\,v^M_{1116}\,p_1^2\,\Big[n\,D - (n-1)\,D_1\Big] - 3\,v^M_{1116}\,p_{12}\,\Big[(n-2)\,D - 3\,(n-1)\,D_1\Big] \\
&+ v^M_{1115}\,p_2^2\,\Big[2\,(n-2)\,D - 3\,(n-1)\,D_1\Big] + 9\,v^M_{1115}\,p_1^2\,\Big[n\,D - (n-1)\,D_1\Big] \\
&- 2\,v^M_{1115}\,p_{12}\,\Big[(n-2)\,D - 3\,(n-1)\,D_1\Big] + v^M_{1113}\,p_1^2\,\Big[n\,D - (n-1)\,D_1\Big] \\
&- 3\,v^M_{1114}\,p_1^2\,\Big[n\,D - (n-1)\,D_1\Big] + v^M_{1114}\,p_{12}\,\Big[(n-2)\,D - 3\,(n-1)\,D_1\Big], \\
&(n+2)\,p_2^2\,V^M(p_1,p_1,p_1\,|\,0) = 3\,v^M_{1111}\,p_1^2\,D_1 + 3\,v^M_{1112}\,p_{12}\,D_1 - 11\,v^M_{1116}\,p_1^4\,D_1\,(n-1) \\
&+ v^M_{1116}\,\Big[3\,n\,D\,D_1 + (n+2)\,D\,D_2 - (n-1)\,D_1\,(D_2 - 9\,D_3 + 3\,D_1)\Big] \\
&- 9\,v^M_{1115}\,p_1^4\,D_1\,(n-1) + 3\,v^M_{1115}\,D_1\,\Big[n\,D + (n-1)\,(2\,D_3 - D_1)\Big] \\
&- (n-1)\,\Big[v^M_{1113}\,p_1^4 - 3\,v^M_{1114}\,p_1^4 + 3\,v^M_{1114}\,D_3\Big]\,D_1 \\
&(n+2)\,p_1^4\,V^M(p_2,p_2,p_2\,|\,0) = \Big[3\,v^M_{1111}\,D_3 + 3\,v^M_{1112}\,D_1 - v^M_{1116}\,p_2^2\,D_1\,(n-1) - 3\,v^M_{1116}\,p_{12}\,D_1\,(n-1)\Big]\,D_1 \\
&- v^M_{1116}\,p_1^2\,\Big[9\,n\,D\,D_1 - 11\,(n+2)\,D\,D_3 + (n-1)\,D_1\,(11\,D_3 - 9\,D_1)\Big] \\
&- 3\,v^M_{1115}\,p_1^2\,\Big[2\,n\,D\,D_1 - 3\,(n+2)\,D\,D_3 + (n-1)\,D_1\,(3\,D_3 - 2\,D_1)\Big] \\
&- 3\,v^M_{1115}\,p_{12}\,D_1^2\,(n-1) + v^M_{1113}\,p_1^2\,D_3\,\Big[(n+2)\,D - (n-1)\,D_1\Big] \\
&+ 3\,v^M_{1114}\,p_1^2\,\Big[n\,D\,D_1 - (n+2)\,D\,D_3 + (n-1)\,D_1\,(D_3 - D_1)\Big].
\end{aligned} \tag{429}
$$

We still need the explicit form of the form factors which requires generalized scalars of Eq.(418):

$$
\begin{pmatrix} V^M_{2221} \\ V^M_{2222} \\ V^M_{2223} \\ V^M_{2224} \\ V^M_{2225} \\ V^M_{2226} \end{pmatrix}
=
\begin{pmatrix}
n+2 & 0 & 2\,p_{12} & p_2^2 & p_1^2 & 0 \\
2 & 0 & p_{12} & 0 & p_1^2 & 0 \\
0 & 1 & p_1^2 & p_{12} & 0 & 0 \\
0 & 0 & 0 & p_1^2 & 0 & p_{12} \\
p_1^2 & p_{12} & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & p_{12} & 0 & p_2^2
\end{pmatrix}^{-1}
\begin{pmatrix} v^M_{2221} \\ v^M_{2223} \\ v^M_{2225} \\ v^M_{2224} \\ v^M_{2226} \\ v^M_{2228} \end{pmatrix}
$$

$$
\begin{pmatrix} V^M_{1221} \\ V^M_{1222} \\ V^M_{1223} \\ V^M_{1224} \\ V^M_{1225} \\ V^M_{1226} \end{pmatrix}
=
\begin{pmatrix}
2 & 0 & p_{12} & 0 & p_1^2 & 0 \\
0 & 1 & p_1^2 & p_{12} & 0 & 0 \\
0 & 0 & 0 & p_1^2 & 0 & p_{12} \\
p_1^2 & p_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & p_{12} & 0 & p_{12} & 0 \\
p_{12} & p_2^2 & 0 & 0 & 0 & 0
\end{pmatrix}^{-1}
\begin{pmatrix} v^M_{1223} \\ v^M_{1225} \\ v^M_{1224} \\ v^M_{1226} \\ v^M_{1227} \\ v^M_{12210} \end{pmatrix}
$$

$$
\begin{pmatrix} V^M_{1121} \\ V^M_{1122} \\ V^M_{1123} \\ V^M_{1124} \\ V^M_{1125} \\ V^M_{1126} \\ V^M_{1127} \\ V^M_{1128} \end{pmatrix} = \begin{pmatrix} 0 & 2+n & p_1^2 & 2\,p_{12} & 0 & p_2^2 & 0 & 2 \\ 2 & 0 & p_{12} & 0 & p_1^2 & 0 & 2 & 0 \\ 0 & 0 & 0 & p_1^2 & 0 & p_{12} & 0 & 0 \\ 0 & 0 & p_2^2 & 0 & p_{12} & 0 & 0 & 0 \\ p_1^2 & p_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & p_{12} & p_2^2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & p_{12} & 0 & p_2^2 & 0 & 2 \\ p_{12} & p_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} v^M_{1122} \\ v^M_{1123} \\ v^M_{1124} \\ v^M_{1127} \\ v^M_{1126} \\ v^M_{1129} \\ v^M_{1128} \\ v^M_{11210} \end{pmatrix}
$$

For the 111 group we have

$$
\begin{aligned}
(n+2)\,V^M_{1111} &= v^M_{1113}\,p_1^2 + v^M_{1114}\,(-3\,p_1^2 + 2\,p_{12}) + v^M_{1115}\,(9\,p_1^2 - 4\,p_{12} + p_2^2) \\
&\quad + v^M_{1116}\,(11\,p_1^2 - 6\,p_{12} + p_2^2) + v^M_{1111}, \\
(n+2)\,V^M_{1112} &= v^M_{1114}\,p_1^2 + 2\,v^M_{1115}\,(-p_1^2 + p_{12}) + v^M_{1116}\,(-3\,p_1^2 + 2\,p_{12} + p_2^2) + v^M_{1112},
\end{aligned}
\tag{430}
$$

$$
\begin{aligned}
V^M_{1113} &= -v^M_{1114} + 2\,v^M_{1115} + 3\,v^M_{1116}, \qquad V^M_{1114} = -v^M_{1115} - v^M_{1116}, \\
V^M_{1115} &= -v^M_{1113} + 3\,v^M_{1114} - 9\,v^M_{1115} - 11\,v^M_{1116}, \qquad V^M_{1116} = -v^M_{1116}.
\end{aligned}
\tag{431}
$$

B.8.4 The V^K family

Most of the fully saturated rank three tensors in this family can be trivially obtained from the partial contractions of Eq.(412) and evaluated with the help of Eqs.(419)–(422). Some of them, however, correspond to contractions leading to irreducible scalar products and, therefore, they require an explicit solution for the form factors. The latter are given in the following list where we use shorthand notation, $P_i = p_i \cdot P$:

$$
\begin{aligned}
&P^2\,P_2\,V^K(p_1,p_1,p_2\,|\,0) = v^K_{1111}\,p_{12}\Big[D\,(P^2 + p_{12}) + p_{12}\,(P^2\,p_{12} + P_1^2)\Big] \\
&+ v^K_{1112}\Big[D\,(P_2^2 + p_{12}^2) + p_{12}^2\,(P^2\,p_{12} + P_2^2)\Big] - v^K_{1114}\,D\,P_2\,p_2^2 + v^K_{1115}\,D\,(P_1\,p_{12} + D) \\
&- v^K_{1116}\Big[(n-1)\,(D\,P_2 + P^2\,p_{12}^2) + D\,p_{12}\Big] + V^K_{1111}\,D^2\,(n-2),
\end{aligned}
$$

$$
\begin{aligned}
&P^2\,P_2\,V^K(p_2,p_2,p_1\,|\,0) = v^K_{1111}\,p_{12}^2\,(D + P^2\,p_{12} + P_2^2) + v^K_{1112}\,P_2\Big[D\,(p_{12} - P_2) + P_2^2\,p_{12}\Big] - V^K_{1111}\,D^2\,(n-2), \\
&- v^K_{1114}\,D\,P_2\,(p_{12} - P_2) + v^K_{1115}\,D\,P_2\,(p_{12} + P_2) + v^K_{1116}\Big[(n-1)\,P_2\,(D - P_2\,p_{12}) - D\,(n-2)\,p_{12}\Big],
\end{aligned}
$$

$$
\begin{aligned}
&P^2\,P_2\,V^K(p_1,p_1,p_1\,|\,0) = v^K_{1111}\,(D + P_1\,p_{12})\,(p_1^2 + D) - V^K_{1111}\,D^2\,(n-2) \\
&+ v^K_{1112}\Big[D\,(D + 2\,P^2\,p_{12} + p_{12}^2) + p_{12}^2\,(P^2\,p_{12} + P_1^2)\Big] - v^K_{1114}\,D\,(2\,P_2\,p_{12} + P_1\,p_{12} + D) \\
&- v^K_{1115}\,D\,(P_1\,p_{12} + D) - v^K_{1116}\Big[(n-1)\,(D\,P_1 - P_1^2\,p_{12}) + D\,(n-2)\,p_{12}\Big],
\end{aligned}
$$

$$
\begin{aligned}
&P^2\,P_2\,p_1^2\,V^K(p_2,p_2,p_2\,|\,0) = v^K_{1111}\,p_{12}^3\,(D + P^2\,p_{12} + P_2^2) + v^K_{1112}\,P_2\Big[p_{12}^2\,(D + P_2^2) + D\,(D - P_2\,p_{12})\Big] \\
&- v^K_{1114}\,D\,P_2\,(p_{12}^2 - P_2^2) + v^K_{1115}\,D\,P_2\,(p_{12}^2 + 2\,P_2\,p_{12} - D) \\
&+ v^K_{1116}\Big[D\,n\,P_2\,p_{12} - (n-2)\,D\,(p_{12}^2 + D) + D\,P_2^2 - (n-1)\,P_2^2\,p_{12}^2\Big] + V^K_{1111}\,D^2\,(n-2)\,p_1^2.
\end{aligned}
\tag{432}
$$

For some special purpose, one may need to have direct access to the explicit expressions of the form factors, or of the uncontracted tensors, which is the same. Here is their solution:

$$
\begin{pmatrix} V^K_{2221} \\ V^K_{2222} \end{pmatrix} = G^{-1} \begin{pmatrix} v^K_{2226} \\ v^K_{22210} \end{pmatrix}, \qquad
\begin{pmatrix} V^K_{2223} \\ V^K_{2224} \\ V^K_{2225} \\ V^K_{2226} \end{pmatrix} = \begin{pmatrix} 2\,p_{12} & p_2^2 & p_1^2 & 0 \\ p_{12} & 0 & p_1^2 & 0 \\ p_1^2 & p_{12} & 0 & 0 \\ 0 & p_1^2 & 0 & p_{12} \end{pmatrix}^{-1} \begin{pmatrix} v^K_{2221} - (n+2)\,V^K_{2221} \\ v^K_{2223} - 2\,V^K_{2221} \\ v^K_{2225} - V^K_{2222} \\ v^K_{2224} \end{pmatrix}
$$

$$V^{K}_{1111} = -\omega^4 \Big[V_K^{1,1|1,1,3|2} + V_K^{1,1|1,2,2|2} + V_K^{1,1|1,3,1|2} + \frac{1}{2} V_K^{1,1|2,1,2|2} + \frac{1}{2} V_K^{1,1|2,2,1|2}$$
$$+ V_K^{1,1|1,1,2|3} + V_K^{1,1|1,2,1|3} + V_K^{1,2|1,1,3|1} + V_K^{1,2|1,2,2|1} + V_K^{1,2|1,3,1|1} + V_K^{1,2|2,1,2|1}$$
$$+ V_K^{1,2|2,2,1|1} + V_K^{1,2|3,1,1|1} + V_K^{1,2|1,1,2|2} + V_K^{1,2|1,2,1|2} + V_K^{1,2|2,1,1|2} + V_K^{1,2|1,1,1|3} \Big],$$

$$V^{K}_{1112} = \frac{1}{p_2 P} \Big[p_1 \cdot P \, V^{K}_{1111} + v^{K}_{1116} \Big]$$

$$\begin{pmatrix} V^K_{1113} \\ V^K_{1114} \\ V^K_{1115} \\ V^K_{1116} \end{pmatrix} = \begin{pmatrix} 2\,p_{12} & p_2^2 & p_1^2 & 0 \\ p_1^2 & 2\,p_{12} & 0 & p_2^2 \\ p_1 \cdot P & p_2 \cdot P & 0 & 0 \\ 0 & p_1 \cdot P & 0 & p_2 \cdot P \end{pmatrix}^{-1} \begin{pmatrix} v^K_{1111} - (n+2)\,V^K_{1111} \\ v^K_{1113} - (n+2)\,V^K_{1112} \\ v^K_{1115} - V^K_{1111} \\ v^K_{1114} - 2\,V^K_{1112} \end{pmatrix}$$

Note that for the 111 group one form factor must be written in terms of generalized functions. This is a typical aspect of the procedure where, sometimes, the equations that one obtains are not all linearly independent.

$$\begin{pmatrix} V^K_{1123} \\ V^K_{1124} \end{pmatrix} = G^{-1} \begin{pmatrix} v^K_{1127} \\ v^K_{11215} \end{pmatrix}, \tag{433}$$

$$\begin{pmatrix} V^K_{1121} \\ V^K_{1122} \\ V^K_{1125} \\ V^K_{1126} \\ V^K_{1127} \\ V^K_{1128} \\ V^K_{1129} \\ V^K_{11210} \end{pmatrix} = \begin{pmatrix} 2 & 0 & p_1^2 & 0 & 2\,p_{12} & 0 & 0 & p_2^2 \\ 1 & 0 & 0 & 0 & 0 & p_1 \cdot P & p_1 \cdot P & 0 \\ 0 & 1 & 0 & 0 & p_1 \cdot P & 0 & 0 & p_2 \cdot P \\ p_1 \cdot P & p_2 \cdot P & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & p_1^2 & 0 & 0 & 0 & p_{12} & 0 \\ 0 & 0 & 0 & p_{12} & 0 & 0 & 0 & p_1^2 \\ 1 & 0 & 0 & 0 & p_1^2 & p_{12} & 0 & 0 \\ 0 & 0 & p_{12} & 0 & 0 & 0 & p_2^2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} v^K_{1121} - n\,V^K_{1123} \\ v^K_{11210} - V^K_{1124} \\ v^K_{11211} - V^K_{1123} \\ v^K_{1123} \\ v^K_{1124} \\ v^K_{1125} \\ v^K_{1126} \\ v^K_{11212} \end{pmatrix}$$

$$\begin{pmatrix} V^K_{1221} \\ V^K_{1222} \end{pmatrix} = G^{-1} \begin{pmatrix} v^K_{1227} \\ v^K_{12216} \end{pmatrix},$$

$$\begin{pmatrix} V^K_{1223} \\ V^K_{1224} \\ V^K_{1225} \\ V^K_{1226} \\ V^K_{1227} \\ V^K_{1228} \\ V^K_{1229} \\ V^K_{12210} \end{pmatrix} = \begin{pmatrix} n & 0 & p_1^2 & 0 & 2\,p_{12} & 0 & 0 & p_2^2 \\ 0 & n & 0 & p_2^2 & 0 & 2\,p_{12} & p_1^2 & 0 \\ 1 & 0 & p_1^2 & 0 & p_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1^2 & 0 & 0 & p_{12} \\ 0 & 0 & 0 & p_{12} & 0 & p_1^2 & 0 & 0 \\ 0 & 0 & p_{12} & 0 & p_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_2^2 & p_{12} & 0 \\ 0 & 1 & 0 & p_2^2 & 0 & p_{12} & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} v^K_{1221} - 2\,V^K_{1221} \\ v^K_{1222} - 2\,V^K_{1222} \\ v^K_{1223} - V^K_{1221} \\ v^K_{1225} - V^K_{1222} \\ v^K_{1224} \\ v^K_{12212} \\ v^K_{12215} - V^K_{1221} \\ v^K_{12213} - V^K_{1222} \end{pmatrix}$$

B.8.5 The V^H family

We obtain tensors with saturated indices in the 111 and 222 groups from the corresponding results in the V^M family by replacing v^M_{111i} and v^M_{222i} with v^H_{111i} and v^H_{222i}. Once again there are saturated tensors leading to contractions with irreducible scalar products that require an explicit solution for the form factors. The latter are given in the following list:

$$p_{12}\, p_1^2\, V^H(\mu\,|\,\mu, p_2) = v^H_{1222}\, D\, p_{12} - v^H_{1228}\, D\, p_2^2\, (n-1)$$
$$- v^H_{1226}\, n\, D\, p_{12} - v^H_{12210}\, (n-1)\, (D - p_{12}^2) + v^H_{1223}\, D_3\, p_1^2$$
$$+ v^H_{1224}\, p_{12} \Big[(n-1)\, D + p_{12}^2 \Big] + v^H_{1225} \Big[(n-1)\, D\, p_1^2 + 2\, D_3\, p_{12} \Big] + v^H_{1226}\, D_3,$$
$$p_{12}\, p_1^2\, V^H(p_2\,|\,p_2, p_2) = v^H_{1223}\, D_3\, p_{12} + v^H_{1224}\, (D\, p_2^2 + D_2\, p_{12}) + 2\, v^H_{1225}\, p_{12}\, D_1 + v^H_{1226}\, D_1,$$
$$V^H(p_2\,|\,p_2, p_1) = v^H_{1223}\, D_3 + v^H_{1224}\, D_2 + v^H_{1225}\, (D + 2\, p_{12}^2) + v^H_{1226}\, p_{12},$$

$$
\begin{aligned}
V^{H}(p_2\,|\,p_1,p_1) &= v^{H}_{1223}\,p_1^4 + v^{H}_{1224}\,p_{12}^2 + 2\,v^{H}_{1225}\,D_3 + v^{H}_{1226}\,p_1^2,\\
p_2^2\,V^{H}(\mu,p_1\,|\,\mu) &= v^{H}_{1121}\,D + v^{H}_{11211}\,D_3 + v^{H}_{1123}\,D_2\\
&\quad + v^{H}_{1124}\Big[D\,(n-1)+p_{12}^2\Big] - v^{H}_{1125}\Big[D\,(n-1)-p_{12}^2\Big] + v^{H}_{1126}\,p_{12}^2,\\
V^{H}(\mu,p_2\,|\,\mu) &= v^{H}_{11211}\,p_1^2 + v^{H}_{1123}\,p_2^2 + v^{H}_{1124}\,p_{12} + v^{H}_{1125}\,p_{12} + v^{H}_{1126}\,n,\\
V^{H}(p_2,p_2\,|\,p_1) &= v^{H}_{11211}\,D_3 + v^{H}_{1123}\,D_2 + v^{H}_{1124}\,p_{12}^2 + v^{H}_{1125}\,D_1 + v^{H}_{1126}\,p_{12}.
\end{aligned}
\tag{434}
$$

The form factors are obtained as follows:

$$
V^{H}_{2221} = \frac{1}{p_1^2}\Big[v^{H}_{2225} - p_{12}\,V^{H}_{2222}\Big],\qquad
\begin{pmatrix} V^{H}_{2223}\\ V^{H}_{2224}\\ V^{H}_{2225}\\ V^{H}_{2226} \end{pmatrix}
=
\begin{pmatrix} 2\,p_{12} & p_2^2 & p_1^2 & 0\\ p_{12} & 0 & p_1^2 & 0\\ p_1^2 & p_{12} & 0 & 0\\ 0 & p_1^2 & 0 & p_{12} \end{pmatrix}^{-1}
\begin{pmatrix} v^{H}_{2221} - (n+2)\,V^{H}_{2221}\\ v^{H}_{2223} - 2\,V^{H}_{2221}\\ v^{H}_{2225} - V^{H}_{2222}\\ v^{H}_{2224} \end{pmatrix}
$$

$$
\begin{aligned}
V^{H}_{2222} = \omega^4\Big[&\frac{1}{2}V_{H}^{1,2|1,1|2,2} + V_{H}^{1,2|1,1|3,1} + V_{H}^{1,3|1,1|2,1} - V_{H}^{2,1|1,1|1,3} - \frac{1}{2}V_{H}^{2,1|1,1|2,2} - \frac{1}{2}V_{H}^{2,2|1,1|1,2}\\
&+ \frac{1}{2}V_{H}^{2,2|1,1|2,1} - V_{H}^{3,1|1,1|1,2}\Big],
\end{aligned}
$$

$$
V^{H}_{1112} = \frac{1}{p_2^2}\Big[v^{H}_{1116} - p_{12}\,V^{H}_{1111}\Big],\qquad
\begin{pmatrix} V^{H}_{1113}\\ V^{H}_{1114}\\ V^{H}_{1115}\\ V^{H}_{1116} \end{pmatrix}
=
\begin{pmatrix} p_1^2 & 2\,p_{12} & 0 & p_2^2\\ p_2^2 & 0 & p_{12} & 0\\ p_{12} & p_2^2 & 0 & 0\\ 0 & p_{12} & 0 & p_2^2 \end{pmatrix}^{-1}
\begin{pmatrix} v^{H}_{1112} - (n+2)\,V^{H}_{1112}\\ v^{H}_{1113}\\ v^{H}_{1115} - V^{H}_{1111}\\ v^{H}_{1114} - 2\,V^{H}_{1112} \end{pmatrix}
$$

$$
\begin{aligned}
V^{H}_{1111} = \omega^4\Big[&V_{H}^{1,1|1,3|1,2} + \frac{1}{2}V_{H}^{1,1|2,2|1,2} + V_{H}^{1,1|1,2|1,3} - \frac{1}{2}V_{H}^{1,1|2,2|2,1} - V_{H}^{1,1|3,1|2,1} + \frac{1}{2}V_{H}^{1,1|1,2|2,2}\\
&- \frac{1}{2}V_{H}^{1,1|2,1|2,2} - V_{H}^{1,1|2,1|3,1}\Big],
\end{aligned}
$$

$$
\begin{pmatrix} V^{H}_{1121}\\ V^{H}_{1122}\\ V^{H}_{1123}\\ V^{H}_{1124}\\ V^{H}_{1125}\\ V^{H}_{1126}\\ V^{H}_{1127}\\ V^{H}_{1128} \end{pmatrix}
=
\begin{pmatrix}
2 & 0 & n & 0 & p_1^2 & 0 & 2\,p_{12} & p_2^2\\
2 & 0 & 0 & 0 & p_1^2 & 0 & p_{12} & 0\\
0 & 1 & 0 & 0 & 0 & 0 & p_1^2 & p_{12}\\
0 & 0 & 0 & 0 & 0 & p_{12} & 0 & p_1^2\\
0 & 0 & p_1^2 & p_{12} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & p_{12} & 0 & p_2^2 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & p_{12} & p_2^2\\
p_{12} & p_2^2 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}^{-1}
\begin{pmatrix} v^{H}_{1121}\\ v^{H}_{1123}\\ v^{H}_{1125}\\ v^{H}_{1124}\\ v^{H}_{1126}\\ v^{H}_{1127}\\ v^{H}_{11210}\\ v^{H}_{11211} \end{pmatrix}
$$

$$
\begin{pmatrix} V^{H}_{1221}\\ V^{H}_{1222}\\ V^{H}_{1223}\\ V^{H}_{1224}\\ V^{H}_{1225}\\ V^{H}_{1226}\\ V^{H}_{1227}\\ V^{H}_{1228} \end{pmatrix}
=
\begin{pmatrix}
0 & 2 & 0 & n & 0 & p_2^2 & p_1^2 & 2\,p_{12}\\
1 & 0 & 1 & 0 & p_1^2 & 0 & p_{12} & 0\\
0 & 1 & 0 & 0 & 0 & 0 & p_1^2 & p_{12}\\
0 & 0 & 0 & 1 & 0 & 0 & p_1^2 & p_{12}\\
0 & 0 & 0 & 0 & 0 & p_{12} & 0 & p_1^2\\
p_1^2 & p_{12} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & p_{12} & 0 & p_2^2 & 0\\
0 & 0 & p_{12} & p_2^2 & 0 & 0 & 0 & 0
\end{pmatrix}^{-1}
\begin{pmatrix} v^{H}_{1222}\\ v^{H}_{12211}\\ v^{H}_{1224}\\ v^{H}_{1223}\\ v^{H}_{1226}\\ v^{H}_{1227}\\ v^{H}_{1229}\\ v^{H}_{12210} \end{pmatrix}
$$

C Symmetry properties

Before discussing the symmetry properties of two-loop functions we briefly describe our strategy to generate Feynman diagrams. We use the *GraphShot* code [34], written in FORM [35], which uses the same logic introduced in [36] and has also been applied to one-loop diagrams in [37]. Well-known packages for diagram handling are listed in [38].

The basic algorithm is inspired by an efficient way of accounting for combinatorial factors in diagrams and uses the results of [39].

$\mathcal{G}raph\mathcal{S}hot$ has a table of all the vertices occurring in the Standard Model, which involves the particles $\gamma, \cdots f, \cdots$. Starting from the class of diagrams one wants to evaluate, $\mathcal{G}raph\mathcal{S}hot$ generates all the graphs by inserting every possible combination of propagators $\gamma\gamma, \overline{f}f, f\overline{f}, \cdots$ in the topology, discarding those containing non existing vertices. Combinatorial factors are included for each graph corresponding to the situation where all propagators are scalar and identical. An example is given in Fig. 18. After their

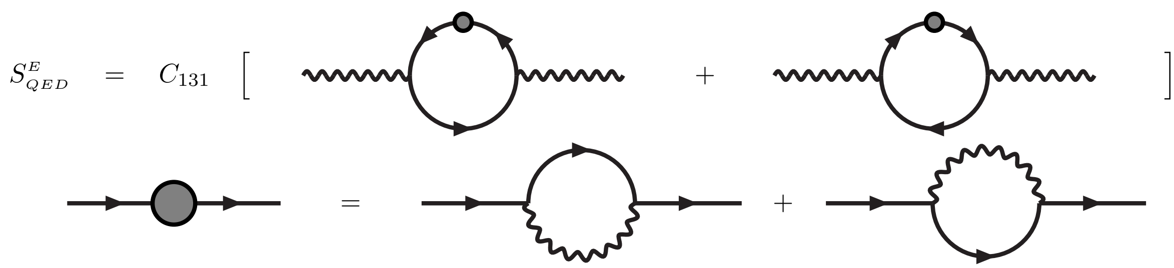

Figure 18: The QED S^E graph as generated by $\mathcal{G}raph\mathcal{S}hot$ [34]. The combinatoric factor C_{131} corresponds to the scalar S^E graph with identical lines and is equal to 1/2.

generation, the diagrams are ready for evaluation or for a check of the corresponding WST identities of the theory. For the latter case we always produce a full scalarization of the result and look for symmetries among the various terms in the result. A typical intermediate output of $\mathcal{G}raph\mathcal{S}hot$ is given in Eq.(301).

Let us give an example of the symmetries: for the S^A family we have the inverse propagators $[1] = q_1^2 + m_1^2, [2]_A = (q_1 - q_2 + p)^2 + m_2^2$, and $[3]_A = q_2^2 + m_3^2$. Consider a set of transformations (where $\oplus$ stands for 'followed by'):

$$\begin{aligned}
\text{a)} &\quad : \quad q_1 \to q_1 + q_2 - p \,\oplus\, q_2 \to -q_2, \\
\text{b)} &\quad : \quad q_1 \leftrightarrow q_2, \\
\text{c)} &\quad : \quad q_2 \to -q_2 + q_1 + p;
\end{aligned} \tag{435}$$

they correspond to symmetries specified by

$$\begin{aligned}
\text{a)} \;\to\; & S^A_0(p, \{m\}_{123}) = S^A_0(-p, \{m\}_{213}), \\
& S^A_1(p, \{m\}_{123}) = -S^A_1(-p, \{m\}_{213}) + S^A_2(-p, \{m\}_{213}) - S^A_0(-p, \{m\}_{213}), \\
& S^A_2(p, \{m\}_{123}) = S^A_2(-p, \{m\}_{213}), \\
\text{b)} \;\to\; & S^A_0(p, \{m\}_{123}) = S^A_0(-p, \{m\}_{321}), \\
& S^A_1(p, \{m\}_{123}) = -S^A_2(-p, \{m\}_{321}), \\
\text{c)} \;\to\; & S^A_0(p, \{m\}_{123}) = S^A_0(p, \{m\}_{132}), \\
& S^A_1(p, \{m\}_{123}) = S^A_1(p, \{m\}_{132}), \\
& S^A_2(p, \{m\}_{123}) = -S^A_2(p, \{m\}_{132}) + S^A_1(p, \{m\}_{132}) + S^A_0(p, \{m\}_{132}).
\end{aligned} \tag{436}$$

For the other two-point functions we recall the conventions: each propagator will be denoted by $[i] = k_i^2 + m_1^2$ and

$$\begin{array}{llllll}
C, & k_1 = q_1, & k_2 = q_1 - q_2, & k_3 = q_2, & k_4 = q_2 + p, & \\
D, & k_1 = q_1, & k_2 = q_1 + p, & k_3 = q_1 - q_2, & k_4 = q_2, & k_5 = q_2 + p, \\
E, & k_1 = q_1, & k_2 = q_1 - q_2, & k_3 = q_2, & k_4 = q_2 + p, & k_5 = q_2.
\end{array}$$

We simply indicate the symmetry property of the scalar configurations; for instance, a change of variables $q_1 \to q_1 + q_2$ followed by $q_2 \to -q_2$ corresponds to a symmetry of the S^C family with respect to the exchange $p \to -p$ and $m_1 \leftrightarrow m_2$:

$$
\begin{aligned}
S^{C} \quad &: \quad q_1 \to q_1 + q_2 \,\oplus\, q_2 \to -q_2, \quad \Rightarrow \quad p \to -p,\; m_1 \leftrightarrow m_2, \\
S^{E} \quad &: \quad m_3 \leftrightarrow m_5, \\
&\quad\;\; q_1 \to q_1 + q_2 \,\oplus\, q_2 \to -q_2, \quad \Rightarrow \quad p \to -p,\; m_1 \leftrightarrow m_2, \\
S^{D} \quad &: \quad q_1 \leftrightarrow q_2, \quad \Rightarrow \quad m_1 \leftrightarrow m_4, \quad m_2 \leftrightarrow m_5, \\
&\quad\;\; q_1 \to q_1 - p\;, \quad q_2 \to q_2 - p, \quad \Rightarrow m_1 \leftrightarrow m_2, \quad m_4 \leftrightarrow m_5.
\end{aligned}
\tag{437}
$$

Finally, let us consider the symmetries of the three-point functions; for the general class V^{1N1} we obtain

$$
\begin{aligned}
V^{1N1} \quad &: \quad q_1 \to q_1 + q_2 \,\oplus\, q_2 \to -q_2, \quad \Rightarrow \quad p_i \to -p_i,\; m_1 \leftrightarrow m_2, \\
V^{M} \quad &: \quad m_3 \leftrightarrow m_6, \\
&\quad\;\; m_4 \leftrightarrow m_5\;\;,\; p_1 \leftrightarrow P\;\;,\; p_2 \to -p_2.
\end{aligned}
\tag{438}
$$

For symmetries in the V^G, V^K and V^H families we have

$$
\begin{aligned}
V^{G} \quad &: \quad q_1 \leftrightarrow q_2, \quad \Rightarrow m_2 \leftrightarrow m_4, \quad m_1 \leftrightarrow m_5, \quad p_1 \leftrightarrow -p_2. \\
V^{K} \quad &: \quad q_i \to q_i - P, \quad \Rightarrow m_1 \leftrightarrow m_2, \quad m_4 \leftrightarrow m_6, \quad p_1 \leftrightarrow -p_2. \\
V^{H} \quad &: \quad q_1 \leftrightarrow q_2, \quad \Rightarrow m_1 \leftrightarrow m_5, \quad m_2 \leftrightarrow m_6, \quad m_3 \leftrightarrow m_4, \quad p_1 \leftrightarrow p_2.
\end{aligned}
\tag{439}
$$

All symmetry properties refer to the scalar configurations.

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